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FINANCIAL REPORT FOR THE SESSION 1889–90 (NOV. 14TH, 1889, TO NOV. 13TH, 1890).

Dr. *CASH ACCOUNT.* **Cr.**

			£.	s.	d.
Balance from 1888–9:—					
General Fund	£43	8 9			
Life Compositions Fund ...	10	10 0			
De Morgan Medal Fund ...	7	19 6			
			61	18	3
Interest on Capital—					
Lord Rayleigh's Fund ...	49	16 8			
Lieut.-Col. Campbell's Fund	8	10 6			
Life Compositions Fund ...	26	10 4			
Invested Surplus Fund ...	13	6 8			
De Morgan Medal Fund ...	2	17 8			
			101	1	10
7 Entrance Fees			7	7	0
102 Subscriptions—					
1 for 1885–6	£1	1 0			
3 for 1886–7	3	3 0			
14 for 1887–8	14	14 0			
45 for 1888–9	47	5 0			
37 for 1889–90	38	17 0			
2 for 1890–1	2	2 0			
			107	2	0
Sales of *Proceedings*...			94	17	4
2 Life Compositions...			21	0	0
Lieut.-Colonel Campbell's Gift			500	0	0
			£893	6	5

			£.	s.	d.
Printing *Proceedings*, &c.			242	12	10
Purchase of Journals, &c.			9	1	6
Binding			3	13	0
Postage, Stationery, and Sundries			16	14	3
Allowance to Member for *Proceedings*			2	8	0
Rent			10	0	0
Coals and Gas			1	10	0
Attendance—					
British Association ...	£1	5 0			
Mr. Stewardson	10	10 0			
			11	15	0
Lieut.-Colonel Campbell's Fund—					
Purchase of £385 London & North Western Railway Consolidated Four per Cent. Guaranteed Stock			499	7	5
Life Compositions Fund—					
Purchase of £32. 3*s.* 7*d.* 2¾ per Cent. Consols			31	10	0
De Morgan Medal Fund—					
Medal	£9	7 6			
Carriage of Dies	0	0 10			
			9	8	4
Balance at Bank—					
General Fund	£53	17 3			
De Morgan Medal Fund	1	8 10			
			55	6	1
			£893	6	5

Audited and found correct,

	£.	s.	d.		£.	s.	d.
Cash at Bank	53	17	3	2 Subscriptions for 1890–1 paid in advance	2	2	0
115 Subscriptions owing—				11 Subscriptions struck off as irrecoverable—			
1 for 1884–5 £1 1 0				1 for 1884–5 £1 1 0			
3 for 1885–6 3 3 0				2 for 1885–6 2 2 0			
3 for 1886–7 3 3 0				2 for 1886–7 2 2 0			
10 for 1887–8 10 10 0				2 for 1887–8 2 2 0			
23 for 1888–9 24 3 0				2 for 1888–9 2 2 0			
75* for 1889–90 78 15 0				2 for 1889–90 2 2 0			
	120	15	0		11	11	0
				Balance	160	19	3
	£174	12	3		£174	12	3

* Viz., 115, the number of Subscribers for 1889–90, less 3 Subscriptions paid in advance in 1888-89, and 37 paid in 1889-90.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	1	8	10	Balance	1	8	10

CAPITAL ACCOUNT.

	Sum Invested. £.	s.	d.		£.	s.	d.	Description of Investment.
General Fund—								
1. Lord Rayleigh's Fund	1000	0	0		870	0	0	Guaranteed Five per Cent. Great Indian Peninsula Railway Stock.
2. Lieut.-Colonel Campbell's Fund	499	7	5		385	0	0	Guaranteed Four per Cent. London & North Western Railway Consolidated Stock.
3. Life Compositions Fund	994	0	0		1013	3	3	2¾ per Cent. Consols, Account A.
4. Invested Surplus Fund	500	0	0		496	19	6	„ „ „ „ B.
De Morgan Medal Fund	103	5	3		104	19	8	„ „ „ „ C.

Audited and found correct,

November 25th, 1890. (Signed) GEORGE HEPPEL.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

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VOL. XXI.

NOVEMBER, 1889, TO NOVEMBER, 1890.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXI.

TWENTY-SIXTH SESSION, 1889—1890.

November 14th, 1889.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Sir JAMES COCKLE, F.R.S., Vice-President, in the Chair.

Messrs. W. Hoover, M.A., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio, U.S.A., and C. H. Thompson, B.A., Lecturer in Mathematics at St. David's College, Lampeter, were elected Members of the Society.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by S. Roberts, F.R.S., seconded by Major O'Callaghan, R.A., and carried.

At the request of the Chairman, Mr. G. Heppel, M.A., consented to act as Auditor.

From the Report of the Secretaries, it appeared that the total number of ordinary members since the General Meeting, held November 8th, 1888, had increased from 190 to 198—of whom 84 are compounders.

The Society had lost three members by death, viz.: Mr. W. J Ibbetson, elected Dec. 11th, 1884; Mr. W. H. Mitchell, elected Jan. 13th, 1881; and Mr. E. Temperley, elected March 11th, 1880.

The following communications had been made:—

On Cyclotomic Functions—§ 1, Groups of Totitives of n; § 2, Periods of n^{th} Roots of Unity: Prof. Lloyd Tanner.

On a Theory of Rational Symmetric Functions: Captain P. A. Macmahon, R.A.

The Factors and Summation of $1^r + 2^r + \ldots + n^r$: Rev. J. J. Milne.

Raabe's Bernoullians: Mr. J. D. H. Dickson.

Certain Algebraical Results deduced from the Geometry of the Quadrangle and Tetrahedron: Dr. Wolstenholme.

On a certain Atomic Hypothesis: Prof. K. Pearson.

On Deep-Water Waves resulting from a Limited Original Disturbance: Prof. W. Burnside.

A Geometrical Note: Mr. H. M. Taylor.

The Equilibrium of a Thin Elastic Spherical Bowl: Mr. A. E. H. Love.

A Method of Transformation with the aid of Congruences of a Particular Type: Mr. J. Brill.

Notes and Illustrations on a former Paper, "On a Method in the Analysis of Ternary Forms": Mr. J. J. Walker.

Results of Ternary Quadric Operators on Products of Forms of any Orders: Mr. J. J. Walker.

On a Theorem in Combinations: Mr. R. W. D. Christie.

On the Diophantine Equation $y^2 + y'^2 =$ Square: Prof. Cayley.

Sur la transformation des équations algébriques: Signor Brioschi.

On Projective Cyclic Concomitants, or Surface Differential Invariants: Mr. E. B. Elliott.

On Secondary Invariants: Prof. L. J. Rogers.

Remarks upon Algebraical Symmetry, with particular reference to the Theory of Operations, and the Theory of Distributions: Major Macmahon, R.A.

Notes on Plane Curves: (iv.) Involution—Condition of a Cubic and its Hessian; (v.) Figure of a certain Cubic and its Hessian: Mr. J. J. Walker.

On Play *à outrance*: Major Macmahon, R.A.

Some Results in the Elementary Theory of Numbers: Mr. C. Leudesdorf.

The Characteristics of an Asymmetric Optical Instrument: Dr. J. Larmor.

A new Angular and Trigonometrical Notation, with Applications: Mr. H. MacColl.

On the Free Vibrations of an Infinite Plate of Homogeneous Isotropic Elastic Matter: Lord Rayleigh.

Ueber die constanten Factoren der Thetareihen im allgemeinen Falle $p = 3$: Prof. F. Klein.

On the Generalised Equations of Elasticity, and their application to the Theory of Light: Prof. K. Pearson.

On the Reduction of a Complex Quadratic Surd to a Periodic Continued Fraction: Prof. G. B. Mathews.

Construction du Centre de Courbure de la développée de la Courbe de Contour apparent d'une Surface que l'on projette orthogonalement sur un Plan: Prof. Mannheim.

On the Solution in Integers of Equations of the Form $x^3 + y^3 + Az^3 = 0$: Mr. S. Roberts.

On the Concomitants of K-ary Quantics: Mr. W. J. C. Sharp.

On the Motion of an Elastic Solid strained by Extraneous Forces: Signor Betti.

Note on the G-function in an Elliptic Transformation Annihilator: Mr. J. *Griffiths.*

On Cyclotomic Functions—§ iii. The Cyclotomics belonging to the f-nomial periods of the p^{th} Roots of Unity, where p is a prime number: Prof. Lloyd Tanner.

On the Complete Elliptic Integrals K, E, G, I: Dr. Kleiber.

The Square of Euler's Series: Dr. Glaisher.

A Theorem in the Calculus of Linear Partial Differential Operations: Major Macmahon, R.A.

On Crystalline Reflection and Refraction: Mr. A. B. Basset.

On some Rings of Circles connected with a Triangle, and the Circles (Schoute's system) that cut them at equal angles: Mr. W. W. Taylor.

The Figures of the Pippian and Quippian of a Class of Plane Cubics: Mr. J. J. Walker.

Generalization of Buffon's Problem: Dr. Sylvester.

On the small Wave-motions of a Heterogeneous Fluid under Gravity: Prof. W. Burnside.

On the Uniform Deformation in Two Dimensions of a Cylindrical Shell of Finite Thickness, with application to the General Theory of Deformation of Thin Shells: Lord Rayleigh.

The same journals had been subscribed for as in the preceding Session.

The first two volumes of the "Œuvres Complètes de Christiaan Huygens," and the first volume of the "Collected Papers" of Prof. Cayley, F.R.S., had been received.

The following addition had been made to the list of exchanges, viz., with the editors of the "Prace matematyczno-fizycne," Warsaw.

The meeting next proceeded to the election of the new Council.

The Scrutators (Messrs. Heppel and W. J. Curran Sharp) having examined the balloting lists, declared the following gentlemen duly elected:—J. J. Walker, F.R.S., President; Sir James Cockle, Knt., F.R.S., E. B. Elliott, M.A., A. G. Greenhill, F.R.S., Vice-Presidents; A. B. Kempe, F.R.S., Treasurer; M. Jenkins, M.A., R. Tucker, M.A., Secretaries. Other Members of the Council:—A. B. Basset, F.R.S., Professor W. Burnside, M.A., Professor Cayley, Sc.D., F.R.S., J. W. L. Glaisher, Sc.D., F.R.S., J. Hammond, M.A., Joseph Larmor, D.Sc., C. Leudesdorf, M.A., Major P. A. Macmahon, R.A., Samuel Roberts, F.R.S.

The following communications were made:—

Isoscelian Hexagrams: Mr. R. Tucker.

On Euler's ϕ-function: Mr. H. F. Baker and Major P. A. Macmahon, R.A.

On the Extension and Flexure of a Thin Elastic Plane Plate: Mr. A. B. Basset.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XLVI., No. 283.

"Proceedings of the Physical Society of London," Vol. X., Part II.

"Proceedings of the Cambridge Philosophical Society," Vol. VI., Part VI.

"Educational Times," for November, 1889.

"Annales de l'École Polytechnique de Delft," Tome V., 1 and 2 Livraisons.

"Bulletin des Sciences Mathématiques," Tome XIII., Oct. and Nov., 1889.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 4.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 91 and 92.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 1, 2, 3, 4.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo II., No. 11.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXIII., 5me Livraison.

"Nieuw Archief voor Wiskunde," Deel XVI., Stuk 1.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXXIV., Heft 2.

"Societa Reale di Napoli—Atti della Reale Accademia delle Scienze Fisiche e Mathematiche," Vol. III.

"Mathematische und Naturwissenchaftliche Berichte aus Ungarn," Band V. (Juni 1886—Juni 1887.)

"Vierde Rapport van de Huygens-Commissie," 8vo pamphlet; Amsterdam, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 10, 1889.

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*Isoscelian Hexagrams.** *By* R. TUCKER, M.A.

[*Read Nov. 14th*, 1889.]

I. *Positive Hexagrams.*†

1. At the point L in BC (Fig. i.), make the $\angle BLN' = B$, $\angle CLM' = C$; then make $\angle M'NA = A$, $\angle BNL' = B$, $\angle L'MC = C$, and join MN'. Now $\angle M'LN' = A = \angle M'NA$, therefore L, N', N, M' are concyclic; also $\angle CLM' = C = \angle CML'$, therefore L, L', M, M' are concyclic; and $\angle AM'N = 180° - 2A = \angle ML'N$, $\therefore$ L', M, M', N are concyclic;

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* *Cf. Proc.*, Vol. XIX., p. 163.

† When λ corresponds to the point of departure of a positive isoscelian, I call the figure a positive hexagram; and when to that of a negative isoscelian, a negative hexagram.

hence L, L', M, M', N, N' are concyclic, and $\angle AMN' = \angle ANM' = A$.

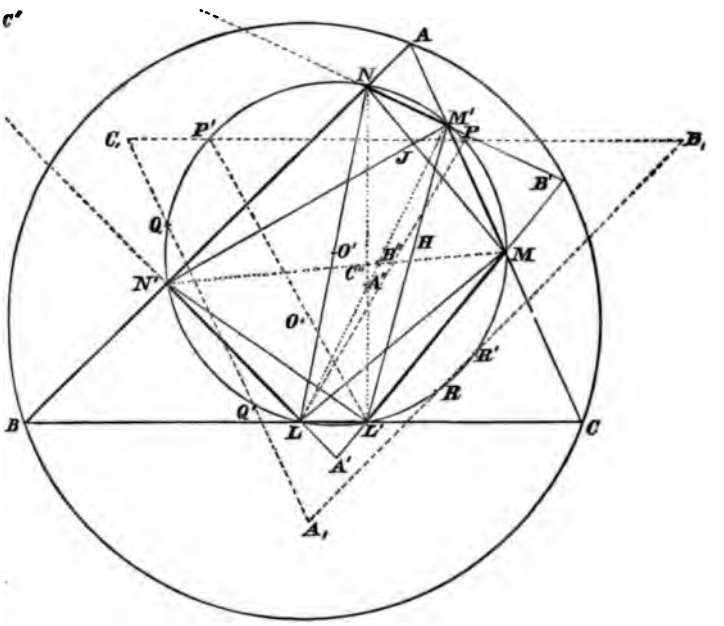


Fig. i.

2. The lines $N'L$, $L'M$, $M'N$ are positive isoscelians, and LM', NL MN' are negative isoscelians. Since

$$\left.\begin{aligned} NL' + L'M &= a, \\ LM' + M'N &= b, \\ MN' + N'L &= c, \end{aligned}\right\}$$

all such isoscelian hexagrams are *isoperimetrical*, and have their perimeters equal to the perimeter of the triangle ABC.

3. Write

$$\left.\begin{aligned} BL = \lambda a, \quad CL &= \lambda' a, \\ CM = \mu b, \quad AM &= \mu' b, \\ AN = \nu c, \quad BN &= \nu' c, \end{aligned}\right\}$$

so that $$\lambda' = 1 = \mu + \mu' = \nu + \nu'.$$

Now $$2BL' \cos B = BN = \nu' c,$$

$$2CL' \cos C = CM = \mu b;$$

whence
$$\left.\begin{aligned} 2a &= \nu' c \sec B + \mu b \sec C, \\ 2b &= \lambda' a \sec C + \nu c \sec A, \\ 2c &= \mu' b \sec A + \lambda a \sec B; \end{aligned}\right\}$$

i.e., if we write

$$p = a\cos A, \quad q = b\cos B, \quad r = c\cos C,$$
$$\mu q = r - p + \lambda p, \quad \mu' q = p + q - r - \lambda p,$$
$$\nu r = r - q + \lambda p, \quad \nu' r = \quad q \quad - \lambda p.$$

4. The trilinear coordinates may be written

$$\left.\begin{matrix} L, & 0, & \lambda' c, & \lambda b \\ M, & \mu c, & 0, & \mu' a \\ N, & \nu' b, & \nu a, & 0 \end{matrix}\right\}, \quad \left.\begin{matrix} L', & 0, & \mu\cos B, & \nu'\cos C \\ M', & \lambda'\cos A, & 0, & \nu\cos C \\ N', & \lambda\cos A, & \mu'\cos B, & 0 \end{matrix}\right\}.$$

5. The equations may be written

to NL', $\quad \nu a\cos C\alpha - \nu' b\cos C\beta + \mu b\cos B\gamma = 0,$

to LM', $\quad \nu c\cos C\alpha + \lambda b\cos A\beta - \lambda' c\cos A\gamma = 0;$

these intersect, in A'', on $c\gamma = bq\beta$; hence AA'', BB'', CC'' cointersec in $\qquad T\,(paa = qb\beta = c\gamma)$;

and the triangle $A''B''C''$,

$$\left.\begin{aligned} A'' &= 2B - C, \\ B'' &= 2C - A, \\ C'' &= 2A - B, \end{aligned}\right\}$$

is similar to the negative in-isoscelian triangle.

6. The equations to LN', ML' are, respectively,

$$\left.\begin{aligned} \mu' b\cos B\alpha - \lambda b\cos A\beta + \lambda' c\cos A\gamma = 0, \\ \mu' a\cos B\alpha + \nu' c\cos C\beta - \mu c\cos B\gamma = 0, \end{aligned}\right\}$$

and these intersect, in A', on $bq\beta = c\gamma$; hence AA', BB', CC' cointersect in T, and the triangle $A'B'C'$,

$$\left.\begin{aligned} A' &= 2C - B, \\ B' &= 2A - C, \\ C' &= 2B - A, \end{aligned}\right\}$$

is similar to the positive in-isoscelian triangle.

The triangles ABC, $A'B'C'$, $A''B''C''$ are then in perspective, with T for their centre of perspective. The point T is the isotomic of the circumcentre.

7. The respective axes of perspective are

$$(ABC,\ A'B'C'),\ \mu'q\,.\,a\alpha+\nu'r\,.\,b\beta+\lambda'p\,.\,c\gamma=0,$$

$$(ABC,\ A''B''C''),\ \nu r\,.\,a\alpha+\lambda p\,.\,b\beta+\mu q\,.\,c\gamma=0,$$

$$(A'B'C',\ A''B''C'')\ (\mu'q-\nu r)\,a\alpha+(\nu'r-\lambda p)\,b\beta+(\lambda'p-\mu q)\,c\gamma=0.$$

8. From § 3, we have the relation

$$0=\begin{vmatrix} -2, & \mu\sec C, & \nu'\sec B \\ \lambda'\sec C, & -2, & \nu\sec A \\ \lambda\sec B, & \mu'\sec A, & -2 \end{vmatrix},$$

whence

$$(\lambda\mu\nu+\lambda'\mu'\nu')\sec A\sec B\sec C+2\mu'\nu\sec^2 A+2\nu'\lambda\sec^2 B+2\lambda'\mu\sec C=8.$$

Now the triangle $L'M'N'$, which is similar to ABC,

$$=\Delta\left[1-\frac{\lambda'\mu}{4\cos^2 C}-\frac{\mu'\nu}{4\cos^2 A}-\frac{\nu'\lambda}{4\cos^2 B}\right]$$

$$=\Delta\,(\lambda\mu\nu+\lambda'\mu'\nu')\sec A\sec B\sec C/8.$$

But, if ρ be the hexagram radius, then triangle $L'M'N'=\rho^2\Delta/R^2$;

therefore $$8\rho^2\cos A\cos B\cos C=R^2\,(\lambda\mu\nu+\lambda'\mu'\nu').$$

The triangle $LMN=\Delta\left\{1-\mu'\nu-\nu'\lambda-\lambda'\mu\right\}=\Delta\,(\lambda\mu\nu+\lambda'\mu'\nu')$,

and it is similar to the pedal triangle of ABC.

9. Let the bisectors of angles $AM'N$, $AN'M$ meet in P; then, since these lines are perpendicular to AB, AC, respectively,

$$\angle M'PN'=\angle A,$$

therefore P is a point on the "H." circle.

The equations to PN', PM' are

$$\mu'a\cos B\alpha-\lambda a\cos A\beta-(\mu+1)\,c\cos B\gamma=0,$$

$$-\nu a\cos C\alpha+(\nu'+1)\,b\cos C\beta+\lambda'a\cos A\gamma=0;$$

these intersect in

$$\frac{a\cos A}{\cos B\cos C\cos(B-C)}=\frac{\beta}{\cos B\cos C+\lambda\sin C\sin A}$$

$$=\frac{\gamma}{\cos B\cos(B-C)-\lambda\sin A\sin B}=\frac{b\beta+c\gamma}{a\cos B\cos C},$$

therefore $$\frac{aa}{\cos(B-C)} = \frac{b\beta+c\gamma}{\cos A} = \frac{\Delta}{\sin B \sin C},$$

i.e., the locus is a straight line B_1C_1, parallel to BC, and bisecting the distance between A and the orthocentre (H).

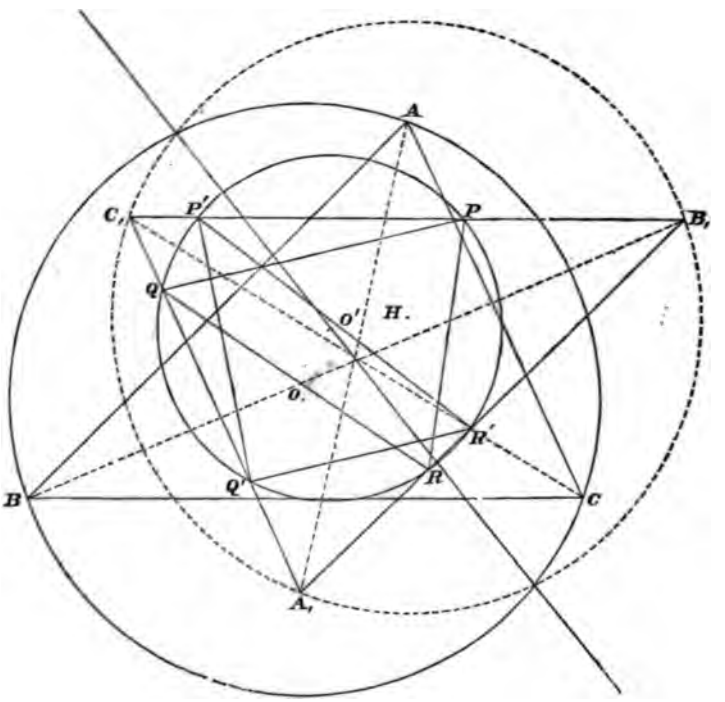


Fig. ii.

Hence, completing the triangle $A_1B_1C_1$, we see that the loci of P, (Q, R), are the lines determined by the sides of this triangle, which we propose to call the *Director* triangle.

P, Q, R are, of course, the mid-points of the arcs MN, NL, LM, and are, further, the orthocentres of the triangles $AM'N'$, $BN'L'$, $CL'M'$.

10. The triangle PQR is similar to ABC, and therefore $= L'M'N'$.

11. By §9, the circle ANM has P for its centre. If we put

$$C \equiv a\beta\gamma + \ldots + \ldots, \quad L \equiv aa + \ldots + \ldots,$$

the circles AMN, BNL, CLM are given by

$$a \,.\, C = L(c\nu'\beta + b\mu\gamma) \quad \text{.....(i.)},$$

$$b \,.\, C = L(a\lambda'\gamma + c\nu a) \quad \text{.....(ii.)},$$

$$c \,.\, C = L(b\mu' a + a\lambda\beta) \quad \text{.....(iii.)},$$

of which the radical axes are, (i., ii.), (ii., iii.),

$$\nu caa - \nu' bc\beta + (a^2\lambda' - b^2\mu)\,\gamma = 0,$$

$$(b^2\mu' - c^2\nu)\,a + \lambda ab\beta - \lambda' ca\gamma = 0;$$

these intersect in $a \cos A = \gamma \cos C$,

hence the radical centre is H.

The straight lines PL, QM, RN consequently pass through H, and the sides PQ, QR, RP of PQR are perpendicular to HN, HL, HM; hence H is the orthocentre of PQR, and the centre of perspective of PQR and LMN.

12. If O be the circumcentre, we have

$$\tan OL'B = 2R \cos A \cos C / (a \cos C - \mu b),$$

and $$\tan HLC = 2R \cos B \cos C / (\lambda' a - b \cos C).$$

Now (§ 3) $$a \cos B \cos C - \mu q = \lambda' p - b \cos C \cos A,$$

hence $$\angle OL'B = \angle HLC,$$

and $L'O$ meets the "H." circle in the point P', where it is cut by B_1C_1 and so for the points Q', R'. Hence the triangles $P'Q'R'$ $L'M'N'$ have O for their centre of perspective.

13. Further, by § 9, $\angle PLM = \angle PLN$, therefore H is the in-centre of LMN, and it is also readily seen to be the circumcentre of $A_1B_1C_1$.

14. Since

$$\angle PL'C = (90° - A) + (180° - 2C) = 90° + B - C = \angle P'LB,$$

$$\therefore \quad R \cos (B - C) = PL' \sin (90° + B - C) = PL' \cos (B - C),$$

i.e., $PL' = R = P'L$, and so for the other points Q, Q'; R, R'.

Hence, if we take any point P on a side of the director triangle, the "H." circle, through P, can be easily constructed.*

15. We have $R = PL' = 2\rho \sin PP'L' = 2\rho \sin OL'B$,

therefore $$OL' = 2\rho \cos A.$$

---

* A variation in the construction is suggested by § 32, *infra*.

Now, since AO makes, with BC, an angle $= 90° + \widehat{B-C} = \angle PL'C$,

therefore $PL' =$ and is parallel to AO,

therefore $AP =$ and is parallel to $OL' = 2\rho \cos A$.

Similarly, $BQ = 2\rho \cos B$, $CR = 2\rho \cos C$.

Hence $AP \sin APP' = R \cos A = 2\rho \cos A \sin APP'$,

therefore $\sin APP' = R/2\rho = \sin BQQ' = \sin CRR'$.

16. The equations to the circles $AM'N'$, $BN'L'$, $CL'M'$ are

$$2 \cos B \cos C \,.\, C = L(\lambda \cos C\beta + \lambda' \cos B\gamma) \ldots\ldots\ldots\ldots \text{(i.)},$$

$$2 \cos C \cos A \,.\, C = L(\mu \cos A\gamma + \mu' \cos C\alpha) \ldots\ldots\ldots\ldots \text{(ii.)},$$

$$2 \cos A \cos B \,.\, C = L(\nu \cos B\alpha + \nu' \cos A\beta) \ldots\ldots\ldots\ldots \text{(iii.)}.$$

The radical axes of (i., ii.), (ii., iii.), (iii., i.) are

$$\mu' \sec A\alpha - \lambda \sec B\beta + (\mu - \lambda') \sec C\gamma = 0,$$

$$(\nu - \mu') \sec A\alpha + \nu' \sec B\beta - \mu \sec C\gamma = 0,$$

$$-\nu \sec A\alpha + (\lambda - \nu') \sec B\beta + \lambda' \sec C\gamma = 0.$$

The radical centre of these is the circumcentre O.

Now $N'L' = 2\rho \sin BNL' = 2\rho \sin B$,

therefore radius of circle $BN'L'$ (and of the others) is ρ; *i.e.*, these circles are equal to the "H." circle.*

17. Assume the equation to circle LMN to be

$$C = L(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma).$$

We get

$$\left.\begin{aligned} a\lambda\lambda' &= \mu_1\lambda' c + \nu_1\lambda b, \\ b\mu\mu' &= \lambda_1\mu c + \nu_1\mu' a, \\ c\nu\nu' &= \lambda_1\nu' b + \mu_1\nu a; \end{aligned}\right\}$$

whence

$$\lambda_1 \begin{vmatrix} 0, & \lambda' c, & \lambda b \\ \mu c, & 0, & \mu' a \\ \nu' b, & \nu a, & 0 \end{vmatrix} = \begin{vmatrix} \lambda\lambda' a, & \lambda' c, & \lambda b \\ \mu\mu' b, & 0, & \mu' a \\ \nu\nu' c, & \nu a, & 0 \end{vmatrix};$$

i.e.,

$$\lambda_1 abc(\lambda\mu\nu + \lambda'\mu'\nu') = a\mu'\nu(-\lambda\lambda' a^2 + \lambda\mu b^2 + \lambda'\nu' c^2),$$

$$\mu_1 \text{ ,, } \text{ ,, } = b\nu'\lambda(-\mu\mu' b^2 + \mu\nu c^2 + \lambda'\mu' a^2),$$

$$\nu_1 \text{ ,, } \text{ ,, } = c\lambda'\mu(-\nu\nu' c^2 + \nu\lambda a^2 + \mu'\nu' b^2);$$

---

* If O_2 be the centre of the circle, then $O'L'O_2N'$ is a square.

whence $$\frac{\lambda_1 \cos A}{\mu'\nu} = \frac{\mu_1 \cos B}{\nu'\lambda} = \frac{\nu_1 \cos C}{\lambda'\mu}$$

$$= \frac{\cos^2 B - 2\lambda \cos B \sin C \sin A + \lambda^2 \sin^2 A}{\sec A \cos B \cos C\,(\lambda\mu\nu + \lambda'\mu'\nu')}.$$

The radical axis of ABC and LMN is

$$L' \equiv \mu'\nu \sec A \,.\, \alpha + \nu'\lambda \sec B \,.\, \beta + \lambda'\mu \sec C \,.\, \gamma = 0,$$

and the equation to LMN is

$$2\Omega = LL'.$$

18. The projection of OL' on BC

$$= (a \cos C - \mu b)/2 \cos C = (c \cos A \cos B - \lambda p)/2 \cos B \cos C,$$

therefore

$$OL'^2 = R^2 \cos^2 A \sec^2 B \sec^2 C\,(\cos^2 B - 2\lambda \cos B \sin C \sin A + \lambda^2 \sin^2 A);$$

hence (by §15) $$4\rho^2 \cos^2 B \cos^2 C = R^2\,(\ldots\ldots),$$

or $$12\rho^2 \cos^2 A \cos^2 B \cos^2 C$$

$$= R^2 \left\{ \begin{array}{l} \cos^2 A \cos^2 B - 2\lambda \sin A \cos^2 A \cos B \sin C + \lambda^2 \sin^2 A \cos^2 A \\ + \cos^2 B \cos^2 C - 2\mu \sin B \cos^2 B \cos C \sin A + \mu^2 \sin^2 B \cos^2 B \\ + \cos^2 C \cos^2 A - 2\nu \sin C \cos^2 C \cos A \sin B + \nu^2 \sin^2 C \cos^2 C \end{array} \right\}.$$

19. If T_a, T_b, T_c be the tangents to LMN from A, B, C, then

$$T_a^2 = \mu' b \,.\, \nu c/2 \cos A, \text{ \&c.},$$

and $$T_a^2 \,.\, T_b^2 \,.\, T_c^2 = \lambda\lambda' a^2 \,.\, \mu\mu' b^2 \,.\, \nu\nu' c^2/8 \cos A \cos B \cos C$$

$$= (LM' \,.\, MN' \,.\, NL')\,(LN' \,.\, ML' \,.\, NM')/8 \cos A \cos B \cos C,$$

which can be otherwise expressed, since

$$8 \cos A \cos B \cos C = (LM \,.\, MN \,.\, NL)/(L'M' \,.\, M'N' \,.\, N'L').$$

20. If O' be the centre of LMN, then the projection (x say) of BO' on $BC = \frac{1}{2}\left[(\lambda+1)\,a - \mu b/2 \cos C\right]$.

Assume α, β, γ to be trilinear coordinates of O', then

$$\gamma + a \cos B = x \sin B = \sin B\left[q - \lambda\,(p - 2a \cos B \cos C)\right]/4 \cos B \cos C,$$

$$a \sin B = \left[q \cos B + \lambda a \sin B \sin (B - C)\right]/\ldots,$$

whence $$M \equiv \gamma \cos B + a \cos 2B = \lambda a \sin 3B/4 \cos B \ldots\ldots\ldots\ldots \text{(i.)},$$

and $$N \equiv a \cos C + \beta \cos 2C = \mu b \sin 3C/4 \cos C \ldots\ldots\ldots\ldots \text{(ii.)},$$

$$L \equiv \beta \cos A + \gamma \cos 2A = \nu c \sin 3A/4 \cos A \ldots\ldots\ldots\ldots \text{(iii.)}.$$

From (ii.), (iii.), with the aid of §2, we get

$$4\cos A\cos B\cos C\left[N\sin(B-C)/\sin 3C+L\sin(C-A)/\sin 3A\right]$$
$$=-\lambda p\cos C\sin(A-B);$$

hence, making use of (i.), we have

$$L\sin(C-A)\sin 3B\sin 3C+M\sin(A-B)\sin 3C\sin 3A+\ldots=0,$$

wherein the coefficient of $\alpha=\sin 3A\left[\Sigma(\cos 2A)-\Sigma(\cos 4A)\right]$,

therefore locus of O' is

$$\alpha\sin 3A+\beta\sin 3B+\gamma\sin 3C=0.*$$

This straight line is perpendicular to OH, being the radical axis of ABC, $A_1B_1C_1$, and passes through the centre of the "N.P." circle, as it should do, since that circle is a circle of this system.

21. Since $\angle P'LB=90°+B-C$,

and $\angle Q'LB=BLN'-Q'QN'=B-90°+C$,

therefore $\angle P'R'Q'=\angle P'L'Q'=180°-2C$,

therefore $\Delta P'Q'R'$ is similar to the pedal triangle and $=\Delta LMN$;

also, $\angle P'OQ'=\angle L'OM'=180°-C$,

therefore O is the incentre of $P'Q'R'$.

Now $\angle M'L'P'=\angle M'PP'=90°-B$,

therefore O is the orthocentre of $L'M'N'$.

From §13,

$$\rho^2(1-8\cos A\cos B\cos C)=O'O^2=O'H^2=\rho^2-2\rho r_1,$$

if r_1 is the inradius of LMN (or $P'Q'R'$);

hence $r_1=4\rho\cos A\cos B\cos C$.

Also, $Q'L'=PM$ (or PN),

$R'M'=QN$ (or QL),

and $P'N'=RL$ (or RM).

22. Referring to §17, we can write the equation to the radical axis $L'=0$ in the form (multiplying all through by $abc\cos A\cos B\cos C$),

$$a\alpha(p+q-r-\lambda p)(r-q+\lambda p)+b\beta\lambda p(q-\lambda p)$$
$$+c\gamma(1-\lambda)p(r-p+\lambda p)=0,$$

* *Cf. Proc.*, xv., p. 129.

or
$$\begin{aligned}&(p+q-r)(r-q)\,aa+p\,(r-p)\,c\gamma\\&+\lambda p\left[\{p+2\,(q-r)\}\,aa+qb\beta+(2p-r)\,c\gamma\right]\\&-\lambda^2p^2\,(aa+b\beta+c\gamma)=0,\end{aligned}\qquad\begin{cases}\equiv\mathfrak{A}\\\equiv-2\mathfrak{B}\lambda p\\\equiv\mathfrak{C}\lambda^2p^2.\end{cases}$$

Hence, for the envelope, we have $\mathfrak{A}\mathfrak{C}=\mathfrak{B}^2$.

Substituting and reducing, we obtain

$$p^2a^2a^2+q^2b^2\beta^2+r^2c^2\gamma^2+2aab\beta\left[2r\,(p+q-r)-pq\right]$$
$$+2b\beta c\gamma\left[2p\,(q+r-p)-qr\right]+2c\gamma aa\left[2q\,(r+p-q)-rp\right]=0;$$

i.e., $p^2a^2a^2+\ldots+\ldots+2pqa\beta\,(4c^2\cos C-ab)+\ldots+\ldots=0.$

To obtain the envelope of the ellipses, we have merely to subtract from the sinister side of the above result

$$4\,(aa+b\beta+c\gamma)\,\frac{2\,(a\beta\gamma+\ldots)}{aa+\ldots}\,pqr;$$

i.e.,
$$8\,(a\beta\gamma+\ldots+\ldots)\,pqr.$$

Hence the envelope is

$$p^2a^2a^2+\ldots+\ldots-2pq\,ab\,a\beta-\ldots-\ldots=0.$$

This is the conic
$$\sqrt{paa}+\sqrt{qb\beta}+\sqrt{rc\gamma}=0.$$

The centre of this conic is the "N.P." centre, and it touches the sides of ABC in (say) t_1, t_2, t_3, so that At_1, Bt_2, Ct_3 intersect in T. Its foci are O and H, its (semi-minor axis)$^2=2R^2\cos A\cos B\cos C$, and its major axis is R; hence the "N.P." circle is its auxiliary circle.

23. Let λ, λ_1 be values corresponding to centres equidistant from the "N.P." centre; then, §20,

$$4\cos B\cos C\sin B.R\cos(B-C)=2q\cos B+(\lambda+\lambda_1)\,a\sin B\sin(B-C),$$

i.e.,
$$2b\cos B\left[\cos C\cos(B-C)-\cos B\right]=(\lambda+\lambda_1)\,a\sin B\sin(B-C),$$

therefore
$$(\lambda+\lambda_1)\,a=2c\cos B.$$

By § 22, the radical axes are given by

$$\mathfrak{A}-2\mathfrak{B}\lambda p+\mathfrak{C}\lambda^2p^2=0,$$
$$\mathfrak{A}-2\mathfrak{B}\lambda_1p+\mathfrak{C}\lambda_1^2p^2=0,$$

whence
$$a\mathfrak{B}=p\mathfrak{C}\,c\cos B,$$

i.e.,
$$(q-r)\,aa+(r-p)\,b\beta+(p-q)\,c\gamma=0,$$

or the radical axis of any two "conjugate" circles of the system is

OH. Hence locus of radical centre of two "conjugate" circles and ABC is the same straight line.

24. From § 16 we see that $N'L'$ envelopes a parabola touching the sides BC, BA, with O for its focus.

If we take, for the moment, a', b', c' to be the mid-points of the sides BC, CA, AB, then Ob' is the direction of the axis of the curve, and $c'a'$ is the tangent at the vertex; hence the latus rectum $= 4R\cos C\cos A$, and the directrix is the side C_1A_1 of the *director* triangle. Similar results hold for the envelopes of $L'M'$, $M'N'$.

25. The equation to the above parabola in trilinear coordinates is

$$\left[(a\alpha - b\beta)\sin(A-B) + c\gamma\sin C\right]^2 = 4ab\,\alpha\beta\sin 2A\sin 2B,$$

so that the chord of contact is

$$(a\alpha - b\beta)\sin(A-B) + c\gamma\sin C = 0.$$

Referred to BC, BA as axes of x and y, the envelope of $N'L'$, *i.e.*, of

$$x/\nu' c + y/\lambda a = 1/2\cos B,$$

is

$$\sqrt{2x\cos C} + \sqrt{2y\cos A} = \sqrt{b}.$$

Hence, if K, K' are the points of contact, we have

$$BK = b/2\cos C, \quad BK' = b/2\cos A,$$

and therefore equation to KK' is

$$2x\cos C + 2y\cos A = b.$$

The equation to CA is $\dfrac{x}{a} + \dfrac{y}{c} = 1$;

hence CA meets KK' in $a/2$, $c/2$; *i.e.*, CA is bisected.

To find direction of axis, we must join B to the mid-point of KK' $(b/4\cos C,\ b/4\cos A)$; the equation to this line is

$$y\cos A = x\cos C;$$

i.e., it passes through H. If it cuts KK' in K'', then mid-point of BK'' is on the curve.

Since the sides of the *director* triangle are directrices of the parabolas, we obtain, from a tangent property, another proof of § 9.

26. From § 11, we see that the envelope of NL is a parabola touching BC, BA, with H as focus. In this case, if AD, BE, CF are the perpendiculars on BC, CA, AB, DF is the tangent at the vertex, and the latus rectum $= 8R\cos A\cos B\cos C$.

In trilinear coordinates, the equation is

$$\left[aa\,(q-r)+b\beta q-c\gamma\,(p-q)\right]^2=4c\gamma\,aa\,rp\,;$$

and, referred to BC, BA, it is

$$\sqrt{x\cos A}+\sqrt{y\cos C}=\sqrt{q}.$$

If K_1, K_1', are the points of contact, then

$$BK_1=q\sec A,\quad BK_1'=q\sec C\,;$$

hence K_1K_1' makes the same angle with BC that KK' does with BA.

The equation to K_1K_1' is

$$x\cos A+y\cos C=q.$$

The equation to the bisector (from B) of K_1K_1' is

$$y\cos C=x\cos A,$$

and this is a line passing through O; hence BO is the direction of the axis.

27. The $L'N'$ series of envelopes can be written in the form

$$\sqrt{\gamma\cos C}+\sqrt{a\cos A}=\sqrt{R}\sin B\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

$$\sqrt{a\cos A}+\sqrt{\beta\cos B}=\sqrt{R}\sin C\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

$$\sqrt{\beta\cos B}+\sqrt{\gamma\cos C}=\sqrt{R}\sin A\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)};$$

these respectively pass through

$$(X_1)\quad 4Ra/c^2\cos A,\quad 2\beta/c\sin A,\quad 4R\gamma/a^2\cos C,$$

$$(X_2)\quad 4Ra/b^2\cos A,\quad 4R\beta/a^2\cos B,\quad 2\gamma/a\sin B,$$

$$(X_3)\quad 2a/b\sin C,\quad 4R\beta/c^2\cos B,\quad 4R\gamma/c^2\cos A,$$

and AX_1, BX_2, CX_3 cointersect in

$$aa\tan A=b\beta\tan B=c\gamma\tan C=2\Delta\tan\omega.$$

28. The LN series of envelopes touch the sides in (K_c, K_b), (L_a, L_c), (M_b, M_a), say.

The equations to the chords are

$$aa\sin A\cos A+b\beta\cos B\sin(C-A)-c\gamma\cos C\sin(A-B)=0\ldots\text{(i.)},$$

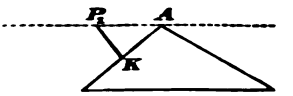
$$-aa\cos A\sin(B-C)+b\beta\sin B\cos B+c\gamma\cos C\sin(A-B)=0\ldots\text{(ii.)},$$

$$aa\cos A\sin(B-C)-b\beta\cos B\sin(C-A)+c\gamma\sin C\cos C=0\ldots\text{(iii.)};$$

(ii.), (iii.) intersect in

$$\frac{aa}{\sin A/\sin(B-C)} = \frac{bb}{1} = \frac{cc}{-1} = \frac{2\Delta}{\sin A \sin(B-C)}$$

i.e., these chords intersect in points (P_1, Q_1, R_1) on lines through A, B, C parallel to the sides:



$$P_1K \text{ (in figure)} = b \sin(B-C),$$

i.e., $$AP_1 = 2R \sin(B-C).$$

The equation to the circle $P_1Q_1R_1$ is

$$C + L\left[4\{a \cos A \sin 2A \sin^2(B-C) + \ldots + \ldots\}/\sin 2A \sin 2B \sin 2C\right] = 0.$$

29. In Fig. iii., S, V, W are the mid-points of the arcs MN', NL',

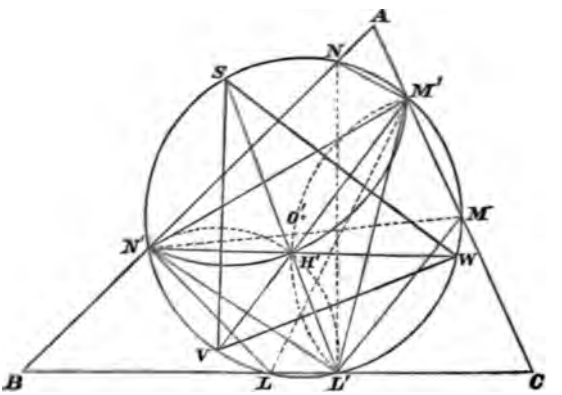


Fig. iii.

$L'M'$; i.e., NS, MS bisect the angles at N' and M; then the angles S, V, W are, respectively,

$$(B+C)/2, \quad (C+A)/2, \quad (A+B)/2,$$

i.e., the triangle is similar to the intriangle of ABC whose circumcircle is the incircle of ABC.

Since $$\angle SWN' = \frac{A}{2},$$

therefore $N'W$ is perpendicular to SV,
$L'S$ " VW,
and $M'V$ " WS.

Let H' be the orthocentre; then circle whose centre is S, and which passes through $M'N'$, also passes through H', and so for the allied circles; whence we see that the sides of SVW bisect $H'L'$, $H'M'$, $H'N'$; hence these distances equal

$$4\rho \sin\frac{B}{2}\sin\frac{C}{2}, \text{ \&c.}$$

Further, H' is clearly the incentre of $L'M'N'$, and also the centre of perspective of this triangle and SVW.

We have, if r_1' be the inradius of $L'M'N'$,

$$\rho^2\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = (O'H')^2 = \rho^2 - 2\rho r_1';$$

therefore
$$r_1' = 4\rho \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

30. If the circle $N'H'M'$ cuts BA in X, then

because
$$\angle H' = 90° + \frac{A}{2},$$

therefore
$$\angle NXM' = 90° - \frac{A}{2},$$

and
$$NX = NM' = AM'.$$

31. Assume
$$\angle N'L'B = \phi,$$

then
$$\frac{\sin(B+\phi)}{\sin\phi} = \frac{BL'}{BN'} = \frac{\nu' c}{\lambda a} = \frac{q-\lambda p}{\lambda a\cos C},$$

therefore
$$\cot\phi = \frac{\cos B - \lambda \sin A \sin C}{\lambda \sin A \cos C};$$

whence, § 18, $\cos\phi = \cos A\,(\cos B - \lambda \sin A \sin C)/D$,

$$\sin\phi = \lambda \sin A \cos C \cos A/D,$$

where D is a symmetrical expression.

Now
$$\alpha\,(\text{of } H') = H'L'\sin\left(\frac{A}{2}+\phi\right)$$

$$= (\S\,29)\ \frac{2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos B\cos C}\,(\lambda'\cos B + \lambda\cos C);$$

hence coordinates are given by

$$\frac{a\alpha}{p(\lambda'\cos B+\lambda\cos C)}=\frac{b\beta}{q(\mu'\cos C+\mu\cos A)}=\frac{c\gamma}{r(\nu'\cos A+\nu\cos C)};$$

or, using § 3,

$$\frac{a\alpha}{p\cos B-\lambda p(\cos B-\cos C)}$$

$$=\frac{b\beta}{\cos A\cos B(2q+b-2c\cos A)-\lambda p(\cos C-\cos A)}$$

$$=\frac{c\gamma}{\cos B(b\cos A+r-q)-\lambda p(\cos A-\cos B)}$$

$$=\frac{a\alpha+b\beta+c\gamma}{2(a+b+c)\cos A\cos B\cos C}=\frac{pa+q\beta+r\gamma}{(a+b+c)\cos A\cos B\cos C};$$

therefore locus of H' is

$$(a-2p)\alpha+(b-2q)\beta+(c-2r)\gamma=0,$$

a straight line which passes through $(b+c)/a$, $(c+a)/b$, $(a+b)/c$,* and is parallel to

$$\alpha\sin 2A+\beta\sin 2B+\gamma\sin 2C=0.$$

32. Since $\angle NN'Q=90°-B=\angle P'PM'$,

therefore $\text{arc } NQ=\text{arc } P'M'$,

and therefore $P'Q=M'N$,

and $\angle QP'C_1=\angle PRQ=\angle C$;

hence $P'QR'PQ'R$ is an equal hexagram.

The negative isoscelians are

$$P'Q\,(=M'N),\quad Q'R\,(=N'L),\quad R'P\,(=L'M);$$

and the positive ones are

$$QR'\,(=M'L),\quad PQ'\,(=N'M),\quad RP'\,(=L'N).$$

33. With centre N', the circle $C_1P'Q'$ passes through O, and the radius $=2\rho\cos C$; and N' is the orthocentre of ΔPQC.

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* This point is the mid-point of TT' (see "A Group of Isosteneans," *Proc.*, Vol. XIX., p. 219).

Since $$\angle PHQ = 180° - C,$$

therefore circle C_1PQ passes through H.

34. Bisect the arcs PQ, QR, RP in W', S', V', respectively;

then $$\Delta S'V'W' = \Delta SVW,$$

because $W'P'$ bisects $\angle QP'C_1$, &c.

Now $$\angle W'V'Q = \angle W'RQ = C/2,$$

therefore $V'Q$ is perpendicular to $S'W'$; hence if H'' be the orthocentre of $S'V'W'$, the circle, centre S', through QR, will pass through H''; this point is also the incentre of ΔPQR.

[The envelopes of PQ, QR, RP, and of $P'Q'$, $Q'R'$, $R'P'$, can be easily obtained by reference to the foregoing results.]

35. We collect here some other results of interest. Suppose

BM, CN intersect in f, $(aa\mu'\nu = b\beta\nu'\mu' = c\gamma\mu\nu)$,

CN, AL „ in g, $(aa\lambda\nu = b\beta\lambda\nu' = c\gamma\lambda'\nu')$,

AL, BM „ in h, $(aa\lambda'\mu' = b\beta\lambda\mu = c\gamma\lambda'\mu)$.

Similarly, let

BM', CN' intersect in f', $\left(\frac{\alpha}{\lambda\lambda'\cos A} = \frac{\beta}{\lambda'\mu'\cos B} = \frac{\gamma}{\lambda\nu\cos C}\right)$,

CN', AL' „ in g', $\left(\frac{\alpha}{\lambda\mu\cos A} = \frac{\beta}{\mu\mu'\cos B} = \frac{\gamma}{\mu'\nu'\cos C}\right)$,

AL', BM' „ in h', $\left(\frac{\alpha}{\nu'\lambda'\cos A} = \frac{\beta}{\mu\nu\cos B} = \frac{\gamma}{\nu\nu'\cos C}\right)$;

then the equations to ff', gg' are

$$aa\left[\nu r\,.\,\lambda\mu\nu - \mu' q\,.\,\lambda'\mu'\nu'\right] + b\beta\left[\lambda p\,.\,\lambda'\mu'\nu' - \nu' r\,.\,\lambda\mu\nu\right] + c\gamma\left[\mu q\,.\,\lambda'\mu'\nu' - \lambda' p\,.\,\lambda\mu\nu\right] = 0,$$

$$aa\left[\nu r\,.\,\lambda'\mu'\nu' - \mu' q\,.\,\lambda\mu\nu\right] + b\beta\left[\lambda p\,.\,\lambda\mu\nu - \nu' r\,.\,\lambda'\mu'\nu'\right] + c\gamma\left[\mu q\,.\,\lambda'\mu'\nu' - \lambda' p\,.\,\lambda\mu\nu\right] = 0;$$

whence we find that ff', gg', hh' cointersect in T, which is therefore the centre of perspective of the triangles fgh, $f'g'h'$.

36. If $MN, M'N'$ meet in f''.

$(NL, N'L')$ " in g''.

and $(LM, L'M')$ " in h''.

then equation to Af'' is

$$\beta \cos C\,(\mu'\nu' q - \nu\lambda' p) = \gamma \cos B\,(\lambda'\mu' p - \mu\nu' r),$$

and Af'', Bg'', Ch'' cointersect in T'':

i.e.,
$$\frac{\alpha}{\cos A\,(\nu'\lambda' r - \lambda\mu q)} = \beta\,(\ldots) = \gamma\,(\ldots).$$

37. For the intersections of $(MN, M'N'; f'')$, $(NL, N'L'; g'')$, $(LM, L'M'; h'')$, we have equation to Ch'',

$$\frac{\alpha}{\cos A\,(\lambda\mu q - \lambda'\nu' r)} = \frac{\beta}{\cos B\,(\lambda'\mu' p - \mu\nu' r)};$$

hence CA, CT', CB, Ch''' form a harmonic pencil.

38. The equation to the circle $A_1B_1C_1$ is

$$4 \sin A \sin B \sin C \,.\, C = L\,(\alpha \sin 3A + \beta \sin 3B + \gamma \sin 3C),$$

and the equation to B_1C_1 is

$$-a\alpha \cos A + (b\beta + c\gamma) \cos (B - C) = 0.$$

39. The conic through intersections of ABC, $A_1B_1C_1$ is

$$\frac{\lambda}{ap \cos (C-A) \cos (A-B)} = \ldots = \ldots$$

$$= \frac{2\lambda'}{bc \cos (B-C) \left[\cos A \cos 2A - \cos (B-C)\right]} = \ldots = \ldots$$

If we call the points of intersection (of the triangles) on BC (1, 2), on CA (3, 4), on AB (5, 6), then (14), (25), (36) cointersect in the "N.P." centre; and if

$$\left.\begin{array}{lll} (35, 46) & \text{intersect in} & \lambda_1 \\ (51, 62) & \text{"} & \text{in } \mu_1 \end{array}\right\}$$

and $(13, 24)$ " in ν_1

then $A\lambda_1$, $B\mu_1$, $C\nu_1$ cointersect in

$$\frac{\alpha}{\sin 2A \cos (B-C)} = \ldots = \ldots,$$

which is the point (β) of Question 9950 of the *Educational Times*.

40. The centroid of $L'M'N'$ is given by

$$a/(\lambda' c\cos B+\lambda b\cos C)\,p=\beta/(\mu' a\cos C+\mu c\cos A)\,q$$
$$=\gamma/(\nu' b\cos A+\nu a\cos B)\,r.$$

Assume $A'\equiv b\cos C-c\cos B,\quad B'\equiv c\cos A-a\cos C,$

$$C'\equiv a\cos B-b\cos A,$$

then the above may be written

$$\frac{a}{pc\cos B+\lambda pA'}=\frac{\beta}{c\cos B\cos A\,(b-2B')+\lambda pB'}$$
$$=\frac{\gamma}{\cos B\,(ar-bC')+\lambda pC'},$$

or $$\frac{A'\beta-B'a}{A'c\cos A\,(b-2B')-B'cp}=\frac{A'\gamma-C'a}{A'\,(ar-bC')-C'pc}$$
$$=\frac{C'\beta-B'\gamma}{C'c\cos A\,(b-2B')-B'(ar-bC')};$$

whence, after reduction, and writing

$$K\equiv a^2+b^2+c^2,$$

we get locus required to be

$$aa\,(3a^2-K)+b\beta\,(3b^2-K)+c\gamma\,(3c^2-K)=0.$$

41. The centroid of LMN is given by

$$\frac{aa}{\mu+\nu'}=\frac{b\beta}{\nu+\lambda'}=\frac{c\gamma}{\lambda+\mu'}=\frac{2\Delta}{3};$$

i.e.,
$$\frac{aaqr}{qa\cos(B-C)-2cr\cos A\cos B-\lambda p\,(q-r)}$$
$$=\frac{b\beta r}{2r-q-\lambda\,(r-p)}=\frac{c\gamma q}{2c\cos A\cos B-\lambda\,(p-q)}$$
$$=\frac{b\beta r+c\gamma q}{r+p+\lambda\,(q-r)}=\frac{r\,(aa+c\gamma)}{q+r+\lambda\,(r-p)}.$$

After reduction, we get for locus

$$aa\,(p^2+q^2+r^2-3qr)+\ldots+\ldots=0.$$

42. To find the symmedian point of $L'M'N'$, we must cut $M'N'$

in a point P, so that

$$M'P : PN' = c^3 : b^3.$$

The coordinates of P will be given by

$$\alpha/(\lambda r+\lambda' q)\cos A, \quad \beta/\mu' r\cos B, \quad \gamma/\nu q\cos C,$$

and the equation to $L'P$ is

$$\alpha\sec A\,(\mu'\nu' r-\mu\nu q)-\beta\sec B\,\nu'\,(\lambda r+\lambda' q)+\gamma\sec C\,\mu\,(\lambda r+\lambda' q)=0.$$

Similarly, the symmedian line through M' is

$$\alpha\sec A\,\nu\,(\mu p+\mu' r)+\beta\sec B\,(\nu'\lambda' p-\nu\lambda r)-\gamma\sec C\lambda'\,(\mu p+\mu' r)=0.$$

Eliminating γ, and removing the factor $r(\lambda\mu\nu+\lambda'\mu'\nu')$, we get coordinates of symmedian point to be

$$\frac{\alpha\sec A}{\lambda r+\lambda' q}=\frac{\beta\sec B}{\mu p+\mu' r}=\frac{\gamma\sec C}{\nu q+\nu' p}$$

$$=P=2\Delta/\left[2\,(pq+qr+rp)-(p^2+q^2+r^2)\right]=2\Delta/(-R),\text{ say.}$$

Hence
$$\alpha\alpha=P\left[p\left\{q-\lambda\,(q-r)\right\}\right],$$
$$b\beta=P\left[2pr-p^2-r^2+qr-p\lambda\,(r-p)\right],$$
$$c\gamma=P\left[q\,(p-q+r)-\lambda p\,(p-q)\right];$$

therefore $pa\alpha+qb\beta+rc\gamma=3pqrP=3pqr\,(a\alpha+b\beta+c\gamma)/[\ldots]$,

i.e., $\alpha\alpha\,(pR+3pqr)+\ldots+\ldots=0$, the locus required.

This passes through

$$a\alpha/(q-r)=\ldots=\ldots,$$

and is, hence, parallel to $pa\alpha+qb\beta+rc\gamma=0$.

43. In like manner, to find the symmedian point of the triangle LMN, we cut MN in P (say), so that

$$MP : PN = r^3 : q^3.$$

The coordinates of P will be given by

$$\alpha/bc\,(r^3\nu'+q^3\mu), \quad \beta/ca r^3\nu, \quad \gamma/ab q^3\mu',$$

and the equation to LP is

$$a\alpha\left[\lambda'\mu' q^3-\nu\lambda r^3\right]+b\beta\lambda\left[\nu' r^3+q^3\mu\right]-c\gamma\lambda'\left[\nu' r^3+q^3\mu\right]=0;$$

similarly, the symmedian line through M is

$$-a\alpha\mu'\left[p^3\lambda'+r^3\nu\right]+b\beta\left[\mu'\nu' r^3-\lambda\mu p^3\right]+c\gamma\mu\left[p^3\lambda'+r^3\nu\right]=0.$$

Eliminating γ, and removing the factor $r^2(\lambda\mu\nu+\lambda'\mu'\nu')$, we have

$$\frac{aa}{r^2\nu'+q^2\mu}=\frac{b\beta}{p^2\lambda'+r^2\nu}=\left[\frac{c\gamma}{q^2\mu'+p^2\lambda}=\frac{2\Delta}{p^2+q^2+r^2}\right].$$

These may be written

$$\frac{aa}{q(2r-p)+\lambda p(q-r)}=\frac{b\beta}{p^2+r^2-qr+\lambda p(r-p)}$$

$$=\frac{c\gamma}{q(p+q-r)+\lambda p(p-q)}=\frac{paa+qb\beta+cr\gamma}{pq(2r-p)+q(p^2+r^2-qr)+qr(p+q-r)}$$

$$=\frac{paa+qb\beta+cr\gamma}{3pqr};$$

i.e., $$3pqr(aa+b\beta+c\gamma)=(p^2+q^2+r^2)(paa+qb\beta+rc\gamma);$$

whence $$aa\left[p(p^2+q^2+r^2)-3pqr\right]+\ldots+\ldots=0$$

is the locus of the symmedian point of LMN. This line is also parallel to $paa+\ldots+\ldots=0$, and is therefore parallel to the line in § 42.

II. *Negative Hexagrams.**

44. The construction is made as in I., 1.

45. Write

$$\left.\begin{aligned} BL'&=\lambda a, & CL'&=\lambda' a\\ CM'&=\mu b, & AM'&=\mu' b\\ AN&=\nu c, & BN'&=\nu' c \end{aligned}\right\}.$$

Here λ, λ' may have the same values as in I. 3, but the μ, ν will be different.

Now

$$2\nu' c\cos B=BL,$$
$$2\mu b\cos C=CL;$$

whence

$$\left.\begin{aligned} a&=2\mu b\cos C+2\nu' c\cos B\\ b&=2\nu c\cos A+2\lambda' a\cos C\\ c&=2\lambda a\cos B+2\mu' b\cos A \end{aligned}\right\};$$

* The geometrical properties of the two hexagrams are, of course, identical; but the analytical work is very different. Sometimes results come more easily by the negative hexagram equations.

i.e.,

$$2\mu' b\cos A = c - 2\lambda a\cos B,$$
$$2\nu c\cos A = b - 2a\cos C + 2\lambda a\cos C,$$
$$2\nu' c\cos A = b - 2\lambda a\cos C,$$
$$2\mu b\cos A = 2b\cos A - c + 2\lambda a\cos B.$$

46. The trilinear coordinates may be written

$$\left.\begin{matrix} L', & 0, & \lambda' c, & \lambda b \\ M', & \mu c, & 0, & \mu' a \\ N', & \nu' b, & \nu a, & 0 \end{matrix}\right\};\qquad \left.\begin{matrix} L, & 0, & \mu\cos C, & \nu'\cos B \\ M, & \lambda'\cos C, & 0, & \nu\cos A \\ N, & \lambda\cos B, & \mu'\cos A, & 0 \end{matrix}\right\}.$$

47. From § 45, we have the identical relation

$$0 = \begin{vmatrix} -1, & 2\mu\cos C, & 2\nu'\cos B \\ 2\lambda'\cos C, & -1, & 2\nu\cos A \\ 2\lambda\cos B, & 2\mu'\cos A, & -1 \end{vmatrix},$$

whence

$$8(\lambda\mu\nu + \lambda'\mu'\nu')\cos A\cos B\cos C - 1$$
$$+ 4\{\mu'\nu\cos^2 A + \nu'\lambda\cos^2 B + \lambda'\mu\cos^2 C\} = 0.$$

The triangles $L'M'N'$, LMN are respectively similar, as in § 8, to ABC, and its pedal triangle; and we find

$$\Delta L'M'N' = \Delta(\lambda\mu\nu + \lambda'\mu'\nu') = \rho'^2\Delta/R^2,$$

where ρ' is the hexagram radius. Also,

$$\Delta LMN = 8\Delta(\lambda\mu\nu + \lambda'\mu'\nu')\cos A\cos B\cos C.$$

48. We may note here that for the positive hexagram

$$LN'.ML'.NM' = \lambda\mu\nu\, abc/8\cos A\cos B\cos C,$$
$$LM'.MN'.NL' = \lambda'\mu'\nu'\, abc/8\cos A\cos B\cos C;$$

whereas for the negative hexagram

$$LN'.ML'.NM' = \lambda'\mu'\nu'\, abc,$$
$$LM'.MN'.NL' = \lambda\mu\nu\, abc.$$

49. We give here some results, referring to the corresponding articles for the positive hexagram.

§ 17: $$\frac{\lambda_1}{\mu'\nu\cos A} = \frac{\mu_1}{\nu'\lambda\cos B} = \frac{\nu_1}{\lambda'\mu\cos C} = \frac{1 - 4\lambda\sin^2 A + 4\lambda^2\sin^2 A}{2\cos^2 A(\lambda\mu\nu + \lambda'\mu'\nu')}.$$

The equation to the radical axis is

$$\mu'\nu a\cos A + \nu'\lambda\beta\cos B + \lambda'\mu\gamma\cos C = 0.$$

§ 18: projection $= a(2\lambda-1)/2 = R\sin A(2\lambda-1)$.

$$OL'^2 = R^2(1-4\lambda\sin^2 A+4\lambda^2\sin^2 A) = 4\rho'^2\cos^2 A.$$

§ 20:
$$2a\cos A = -\{R\cos 2B+\lambda a\sin(B-C)\},$$
$$2\beta\cos A = \{R\cos(2A-C)-\lambda a\sin(C-A)\},$$
$$2\gamma\cos A = \{R\cos B-\lambda a\sin(A-B)\};$$

multiplying these respectively by $\sin 3A$, $\sin 3B$, $\sin 3C$, we get

$$a\sin 3A+\ldots+\ldots = 0.$$

§ 23:

$$2R\cos A\cos(B-C) = -2R\cos 2B-(\lambda+\lambda_2)a\sin(B-C),$$

i.e., $$R(\cos 2B+\cos 2C-2\cos 2B) = (\lambda+\lambda_2)a\sin(B-C);$$

therefore $$\lambda+\lambda_2 = 1.$$

NOTE.

[50. Referring to § 9, through P draw a parallel to BC, cutting the circle in P', and the perpendicular (AD) in J. Then

$$\angle PM'L' = \tfrac{1}{2}AM'N+\angle MM'L',$$

and
$$\angle MM'L' = \angle MNL' = C-\angle MNM'$$
$$= C-(AM'N-NPM'),$$

therefore
$$\angle PM'L' = C+NPM'-\tfrac{1}{2}AM'N$$
$$= C+A-90^\circ+NPM' = \angle APP';$$

then $$PL' = 2\rho\sin PM'L' = 2\rho\sin APP',$$

and
$$AJ = AP\sin APP' = PN\sin APP'$$
$$= 2\rho\cos A\sin APP' = PL'\cos A.$$

Again, by § 14, perpendicular from P on $BC = PL'\cos(B-C)$; hence J is the mid-point of AH, and $PL' = R$.

51. Let the normal at a point P of an ellipse cut the axes in G, g, and parallels to the axes, through P, meet them in N, n; then, with the usual notation, if $Pg = r$,

because $$PF\,.\,Pg = AC^2 = a^2,$$

we have $r^2-a^2 = r\,.\,gF = Cg\,.\,gn$ (because C, F, P, n are concyclic);

but since $$CG = e^2\,.\,Pn,$$

therefore $$r^2-a^2 = (Cg)^2/e^2 = d^2/e^2, \text{ say.}$$

Hence, if from any point on the minor axis as centre, with gP as radius, we describe a circle to touch the ellipse, we have the above relation.

Now, if we call the distance between O' and the "N.P." centre d, and put θ for the angle which the line connecting them makes with BC and $2a$, ϵ, for the major axis and eccentricity of the ellipse of § 22, we have

$$2d\cos\theta = BL+BL'-R\{2\sin A+\sin(C-B)\}$$

$$= \frac{1}{2\cos B\cos C}\{2\lambda a\cos B\cos C+b\cos B-\lambda a\cos A\}-R\{\ldots\}$$

$$= \frac{R}{\cos B\cos C}\{2\cos B\cos C-\cos A\}\{\lambda\sin A-\cos B\sin C\};$$

but $$OH\cos\theta = 2\cos B\cos C-\cos A,$$

therefore $$d\cos B\cos C = (\lambda\sin A-\cos B\sin C)\,a\epsilon,$$

i.e., $$d^2\cos^2 B\cos^2 C = a^2\epsilon^2\,.\,\{\lambda^2\sin^2 A-2\lambda\sin A\cos B\sin C$$
$$+\cos^2 B-\cos^2 B\cos^2 C\}$$
$$= a^2\epsilon^2\,.\,\{4\rho^2/R^2-1\}\cos^2 B\cos^2 C, \text{ by § 18,}$$

i.e., $$\frac{d^2}{\epsilon^2} = a^2\left[\frac{4\rho^2}{R^2}-1\right] = \rho^2-a^2;$$

hence, if we take $\rho = r$, we see that the "H." circles touch the in-ellipse of § 22.

From the above result we at once get

$$\rho^2\epsilon^2 = d^2+a^2\epsilon^2 = OO'^2 \text{ (or } O'H^2),^*$$

therefore $$OO' = O'H = \rho\epsilon;$$

and $$2\rho\cos O'OH = 2a = R, \quad cf. \text{ § 15.}$$

52. If we suppose the points L, L' to coincide† so that BC is a tangent to the H. circle, then, if ρ_a be the radius of the circle, we have $$LN' = 2\rho_a\sin B, \quad LM = 2\rho_a\sin 2C;$$

hence $$a = 2\rho_a\{\sin 2B+\sin 2C\},$$

i.e., $$2R\sin A = 4\rho_a\sin A\cos(B-C),$$

---

* This result follows also from §§ 21, 22.

† The reader is requested to draw the figure.

or
$$\left.\begin{aligned} R &= 2\rho_a \cos(B-C), \\ &= 2\rho_b \cos(C-A), \\ &= 2\rho_c \cos(A-B), \end{aligned}\right\}$$

if ρ_b, ρ_c are the radii when the circle touches CA, AB respectively.

The points L, M, N (when M, M'; N, N' coalesce) are readily seen to be the points of contact of the sides with the in-ellipse, so that AL, BM, CN (in this case) cointersect in T.

53. Since
$$BN \,.\, BN' = BL^2 = 4\rho_a^2 \sin^2 2B,$$

and
$$BN' = LN' = 2\rho_a \sin B,$$

therefore
$$BN = 2\rho_a \sin B \,.\, 4\cos^2 B;$$

hence
$$NN' = 2\rho_a \sin 3B \text{ and } MM' = 2\rho_a \sin 3C.$$

Now the angle
$$O'N'A = (\pi - 2C) + (\pi - 2A) - \left(\frac{\pi}{2} - B\right) = 3B - \frac{\pi}{2},$$

and
$$\angle\, O'MA = \frac{3\pi}{2} - 3C;$$

hence coordinates of O' are
$$\rho_a, \quad -\rho_a \cos 3C, \quad -\rho_a \cos 3B.$$

54. The equation to the circle, which passes through
$$N' \{b \sin B,\ a \sin(2C-B),\ 0\}, \quad L(0,\ c^2 \cos C,\ b^2 \cos B),$$
$$M' \{c \sin C,\ 0,\ a \sin(2B-C)\},$$

is
$$a\beta\gamma + \ldots + \ldots = \frac{(a\alpha + \ldots + \ldots)}{4c \sin A \sin B \cos^2(B-C)}$$
$$\times \left[2a\alpha \cos A \sin(2C-B) \sin(2B-C) + \beta b \sin^2 2B + \gamma c \sin^2 2C\right].$$

55. If we project $O'O$ on BC, and on a perpendicular thereto, we get
$$(O'O)^2 = \rho_a^2 \left[\sin 2B \sim \sin 2C)^2 + (1 + \cos 2B + \cos 2C)^2\right]$$
$$= \rho_a^2 \left[3 + 2(\cos 2A + \cos 2B + \cos 2C)\right]$$
$$= \rho_a^2 \left[1 - 8 \cos A \cos B \cos C\right];$$

therefore
$$O'O = e\rho_a, \quad cf. \S\, 51.$$

Again,

$$OL^2 = \left[2\rho_a \cos A \sin (B \sim C)\right]^2 + \left[2\rho_a \cos A \cos (B \sim C)\right]^2$$
$$= 4\rho_a^2 \cos^2 A,$$

therefore $OL : OO' : LO' = 2\cos A : \epsilon : 1$;

and $\angle OLO' = B \sim C$, therefore $\angle OLH = 2(B \sim C)$.

From the above, we have

$$\epsilon \cos OO'L = 1 + \cos 2B + \cos 2C;$$

and $\sin HLC = \sin OLB = R\cos A / 2\rho_a \cos A = \cos (B \sim C)$,

therefore $$\angle HLC = \frac{\pi}{2} - (C \sim B) = \phi.$$

56. If ρ_s is the radius of curvature of the ellipse at L, we have

$$(\alpha\beta\rho_s)^{\frac{2}{3}} \sin^2 \phi = \beta^2;$$

therefore

$$\rho_s = \beta^2 \operatorname{cosec}^3 \phi / \alpha = 4R \cos A \cos B \cos C / \cos^3 (B - C)\big].$$

57. Let Δ' be the area of the triangle $t_1 t_2 t_3$ (§ 22), and put

$$\mathfrak{D} \equiv \cos (A - B) \cos (B - C) \cos (C - A);$$

then
$$\Delta - \Delta' = R^2 \left[\sin^2 2A \sin A \cos (B - C) + \ldots + \ldots\right] / 2\mathfrak{D}$$
$$= R^2 \left[\sin 2A \sin 2B (\sin 2A + \sin 2B) + \ldots + \ldots\right] / 4\mathfrak{D}$$
$$= R^2 \sin A \sin B \sin C$$
$$\times \left[\left\{1 + \cos 2A + \cos 2B + \cos 2(A - B)\right\} + \ldots + \ldots\right] / 2\mathfrak{D}$$
$$= R^2 \sin A \sin B \sin C \left[1 + \cos 2A + \cos 2B + \cos 2C\right.$$
$$\left. + 2\cos (A - B) \cos (B - C) \cos (C - A)\right] / \mathfrak{D}$$
$$= \Delta - R^2 \sin A \sin B \sin C \,(4 \cos A \cos B \cos C) / \mathfrak{D},$$

i.e., $$\Delta' = 2\Delta \cos A \cos B \cos C / \mathfrak{D}.$$

58. From the Geometry of the Ellipse, the joins of A (B, C) and of the "N.P." centre bisect $t_2 t_3$ $(t_3 t_1,\ t_1 t_2)$.

59. If t_1', t_2', t_3' are the mid-points of $t_2 t_3$, $t_3 t_1$, $t_1 t_2$, then equations of $t_1 t_1'$, $t_2 t_2'$ are

$$a\alpha \sin 2A p_1 + b\beta \sin 2B p_2 - c\gamma \sin 2C p_3 = 0,$$
$$-a\alpha \sin 2A q_3 + b\beta \sin 2B q_1 + c\gamma \sin 2C q_2 = 0,$$

where $p_1 \equiv c \sin 2C \cos(A-B) - b \sin 2B \cos(C-A)$,

$p_2 \equiv$ " + "

and similar expressions for q, r; hence we get the coordinates of the centroid of $t_1 t_2 t_3$ to be

$$\alpha/p_2 \cos(B-C) = \beta/q_2 \cos(C-A) = \gamma/r_2 \cos(A-B).$$

We may put p_2 into the form

$$R\left[E + \sin 2B \sin 2C\right],$$

where $E \equiv \sin 2A \sin 2B + \ldots + \ldots$.

60. The circle $t_1 t_2 t_3$ has for its equation

$$C = L(\lambda\alpha + \mu\beta + \nu\gamma),$$

where $\lambda = \dfrac{\begin{vmatrix} 0 & c\sin 2C & b\sin 2B \\ c\sin 2C & 0 & a\sin 2A \\ b\sin 2B & a\sin 2A & 0 \end{vmatrix}}{R\sin^2 2A \begin{vmatrix} \dfrac{\sin 2B \sin 2C}{\cos(B-C)} & c\sin 2C & b\sin 2B \\ \dfrac{\sin 2C}{\cos(C-A)} & 0 & a \\ \dfrac{\sin 2B}{\cos(A-B)} & a & 0 \end{vmatrix}}$;

hence equation is

$$2abc\,\mathfrak{D}\,.\,C = L\left[\begin{matrix} a^2\alpha \cos A\,(\sin B \sin C + 2\cos A \cos 2B \cos 2C) \\ +\ldots \qquad +\ldots \end{matrix}\right],$$

61. If we suppose our "H." circle to touch the sides CA, AB in M, N, then the hexagram is $NLMNL'MN$, *i.e.*, MN is a doubled line. From a consideration of the figure, we see that

$$A = 180° - 2A,$$

i.e., AMN must be an equilateral triangle, and we can further readily prove that M, N are the points t_2, t_3.

The radius of the "H." circle in this case is

$$(\sqrt{3} + 2\sin 2B)/c = 1/\rho = (\sqrt{3} + 2\sin 2C)/b.$$

January, 1890.]

*On Euler's ϕ-Function.** *By* H. F. Baker.

[*Read Nov. 14th*, 1889.]

It is perhaps worth noticing that the theorem on combinations given by Mr. R. W. D. Christie (*Proceedings*, Jan. 10th, 1889) is a particular case of a theorem in the inversion of finite series, of which another case was given by Dedekind (*Crelle*, Bd. 54, p. 25).

Let $f(a_1, a_2, \dots a_k)$ be any quantity, or aggregate of quantities, uniquely determinate from the arguments $a_1, a_2, \dots a_k$ (which may be of any number), and independent of the order of these. Consider the function $F(a_1 \dots a_k)$ defined for all values of k by the equation

$$F(a_1, a_2, \dots a_k) = f(a_1, a_2, \dots a_k) + \sum_1 f(a_2, a_3, \dots a_k)$$
$$+ \sum_2 f(a_3, a_4, \dots a_k) + \dots\dots + \sum_{n-1} f(a_1) + f(0),$$

where $\sum_r$ indicates a summation extending to all selections of $n-r$ of the arguments $a_1, a_2, \dots a_k$.

Then the function

$$F(a_1, a_2, \dots a_n) - \sum_1 F(a_2, \dots a_n) + \sum_2 F(a_3, a_4, \dots a_n) - \dots$$
$$\dots + (-1)^{n-1} \sum_{n-1} F(a_1) + (-1)^n F(0)$$

is, in fact,

$$f(a_1, a_2, \dots a_k).$$

For, suppose each F herein replaced, according to its definition, by functions f. Then the function $f(a_1, a_2, \dots a_k)$ (provided $k \neq n$) will occur once from $F(a_1 \dots a_n)$,

$$\binom{n-k}{1} = \frac{\lfloor n-k}{\lfloor 1 \, \lfloor n-k-1}$$ times from $\sum_1 F(a_2 \dots a_n)$,

$$\binom{n-k}{2} = \frac{\lfloor n-k}{\lfloor 2 \, \lfloor n-k-2}$$ times from $\sum_2 F(a_3, a_4, \dots a_n)$,

and so on, and will thus occur, on the whole,

$$1 - \binom{n-k}{1} + \binom{n-k}{2} - \dots = (1-1)^{n-k} \text{ times},$$

that is, not at all. So that the theorem is obvious.

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* Euler uses $\pi(n)$ not $\phi(n)$. (*Opera Arithmetica*, Fuss, Petropoli, 1849, II., p. 128.)

Corollary I.—By taking

$$f(a_1, a_2, \ldots a_k) = \Sigma \frac{\lfloor m}{\lfloor a_1 \lfloor a_2 \ldots \lfloor a_k} a_1^{a_1} a_2^{a_2} \ldots a_k^{a_k},$$

where $a_1, \ldots a_k$ can have all positive integer values subject to

$$a_1 + a_2 + \ldots + a_k = m,$$

excluding zero values, and putting $f(0) = 0$, we obtain

$$(a_1+a_2+\ldots+a_n)^m - \sum_1 (a_2+a_3+\ldots+a_n)^m + \sum_2 (a_3+a_4+\ldots+a_n)^m - \ldots$$
$$\ldots + (-1)^{m-1} \sum_{n-1} a_1^m = \lfloor n\, a_1 a_2 \ldots a_n \text{ or } 0,$$

according as $m = n$ or is less than n. It is this theorem, when m is less than n, which is given by Mr. Christie. A well-known theorem is obtained when $a_1 = a_2 = \ldots = a_n = 1$.

Corollary II. — Let $a_1, a_2, \ldots a_n$ denote different positive prime numbers, and $[1] = A, B, C, \ldots K$ be any aggregate of positive integers. Also, let $F(a_1, a_2, \ldots a_k)$ denote the aggregate of all the numbers in [1] which are divisible by every one of the primes a_{k+1}, $a_{k+2}, \ldots a_n$, so that $F(0)$ will consist of the numbers in [1] which are divisible by each of $a_1, a_2, \ldots a_n$, and $F(a_1, a_2, \ldots a_n)$ will be those as to which nothing is conditioned, that is, will be [1]. Then the numbers $F(a_1, a_2, \ldots a_k)$ may be divided further into groups. 1st, those which are divisible by $a_{k+1}, a_{k+2}, \ldots a_n$, but by no other of the primes $a_1, a_2, \ldots a_n$. These we denote by $f(a_1, a_2, \ldots a_k)$. 2nd, those which are divisible by $a_{k+1}, a_{k+2}, \ldots a_n$, and also by one other prime, say a_1, but by none other of the primes. These we denote by $f(a_2, a_3, \ldots a_k)$. So we have groups $f(a_1, a_3, \ldots a_k)$, &c. 3rd, and in general, let $f(a_{r_1}, a_{r_2}, \ldots a_{r_t})$ denote the aggregate of numbers in [1] which are not divisible by any of the primes $a_{r_1}, a_{r_2}, \ldots a_{r_t}$, but are divisible by each of the others, so that $f(0)$ will be the numbers divisible by every one of $a_1, a_2, \ldots a_n = F(0)$, and $f(a_1, a_2, \ldots a_n)$ will be those which are prime to $a_1 a_2 \ldots a_n$; then, dividing into groups in the way indicated, we have

$$F(a_1, a_2, \ldots a_k) = f(a_1, a_2, \ldots a_k) + \sum_1 f(a_2, \ldots a_k) + \sum_2 f(a_3, \ldots a_k) + \ldots$$
$$\ldots + \sum_{n-1} f(a_1) + f(0) \ldots\ldots\ldots (\text{I.})$$

for all values of k.

Thus, by the theorem given, we have

$$f(a_1, a_2, \ldots a_n) = F(a_1, \ldots a_n) - \sum_1 F(a_2, \ldots a_n) + \sum_2 F(a_3, a_4, \ldots a_n) - \ldots$$
$$\ldots + (-1)^{n-1} \sum_{n-1} F(a_1) + (-1)^n F(0) \ldots (\text{II.}).$$

And in the particular case when the set [1] consists of

$$1, 2, 3, 4, \ldots N,$$

and

$$N = a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n},$$

the equation (II.) shows that (as is well known) the numbers in [1] which are prime to N can be written down as

$$[1] - \Sigma [a_1] + \Sigma [a_1 a_2] - \ldots + (-1)^{n-1} \Sigma [a_2 a_3 \ldots a_n] + (-1)^n [a_1 a_2 \ldots a_n],$$

$[a_r a_s \ldots]$ denoting the numbers of [1] which are divisible by $a_r a_s \ldots$

$$\left(\text{in particular, that } \phi(N) = N - \Sigma \frac{N}{a_1} + \Sigma \frac{N}{a_1 a_2} - \ldots\right);$$

while equation (I.) gives that decomposition (due to Gauss, *Dis. Arith.*, § 39) of the numbers $[1] = 1, 2, \ldots N$ into sets $[\overline{d}]$, where d denotes in turn all the divisors of N, and $[\overline{d}]$ denotes all the numbers of [1] which have d for the greatest divisor common to them and N (for the divisors of N can be divided into sets according to the number of prime factors $a_1, a_2, \ldots a_n$ which they contain); namely, in this notation the equation (I.) is

$$[1] = [\overline{1}] + \Sigma [\overline{d}] + [\overline{n}],$$

and proves the relation

$$N = \phi(1) + \Sigma \phi(d) + \phi(n),$$

from which equation, therefore, the value of $\phi(N)$ can be deduced,—and this, in fact, it is, which Dedekind notices in the article referred to.

Note.—It should be noticed that equation (I.) is satisfied by putting F equal to any explicit rational integral symmetric function of $a_1 \ldots a_k$. Equation (II.) gives then many theorems in symmetric functions

*On the Extension and Flexure of a Thin Elastic Plane Plate.**

By A. B. Basset, M.A., F.R.S.

[*Read November 14th*, 1889.]

1. The object of every theory of thin elastic plates and shells is to express all the quantities involved, in terms of the displacements of a point on the middle surface of the plate or shell, and their differential coefficients with respect to the coordinates of the undisplaced position of this point; and, in order to do this, it has been usual to assume that the three stresses R, S, and T may be treated as zero, where R is the traction perpendicular to the middle surface of the plate or shell, and S and T are the two shearing stresses which tend to produce rotation about two lines of curvature of the middle surface.

A very slight consideration of the subject is sufficient to show that R, S, T cannot be rigorously zero. The assumption that S and T are zero would involve the absence of any normal shearing stress across a section of the plate, whereas it must be obvious to everybody that a thin plate of metal, such as a worn half-sovereign, is capable of sustaining a very considerable normal shearing stress. The assumption that R is zero is also immediately seen to be incorrect, if the plate is under the action of external pressure; for, if the pressures on the two surfaces be denoted by Π_1, Π_2, the value of R, as we pass through the plate from one surface to the other, must vary from $-\Pi_1$ to $-\Pi_2$. Even if there were no external pressures, it is most improbable that R could be zero throughout the substance of the plate or shell. For, suppose the shell to be cylindrical, and to be divided by an imaginary surface parallel to its middle surface; then, if R were zero, the two portions on either side of the imaginary surface would be incapable of exerting any action in the nature of a pressure or tension upon one another. If, therefore, the imaginary surface were a real surface of separation dividing the shell into two separate portions, the two portions would, on the assumption that $R=0$, always remain

* [In the paper as read, certain remarks were made concerning the theory of cylindrical and spherical shells; these are now withdrawn, as I have since dealt with this subject in a paper, an Abstract of which is published in the *Proc. Royal Society*, Vol. xlvii., p. 45.—Feb. 19th, 1890.]

in contact, whether the other were in equilibrium or in motion. But if the shell were bent in such a manner that its curvature were diminished, the most probable supposition would be that the two divided portions would separate; whilst, if the bending increased the curvature, we should anticipate that each portion would exert a pressure upon the other. It can, however, be rigorously proved that, if R is of the order of the square of the thickness, then S and T are of the same order; and that, whenever this is the case, these stresses may be treated as zero, in calculating the expression for the potential energy, inasmuch as they give rise to terms of a higher order than the cube of the thickness. The first point, therefore, to ascertain is, in what way does R depend upon the thickness?

For a plane plate of infinite extent, this question has been dealt with by Lord Rayleigh;* for, by means of equation (33) of the paper referred to below, it can easily be shown that, in this case, R is of the order of the square of the thickness, *provided the surfaces of the plate are not subjected to external pressure or tangential stress.* Lord Rayleigh's investigation, of course, does not rigorously apply to a plate of finite dimensions, but it seems most improbable that a similar result would not hold good in the case of a finite plate, and *this will be assumed to be the case.* It will, moreover, appear that this assumption is justified by the results to which it leads.

We have now all the materials we require for solving the problem of the equilibrium and motion of a thin plane plate of finite dimensions, which we shall proceed to discuss. It will be found that the equation of motion and the boundary conditions for the flexural vibrations are the same as those obtained by other writers, but that the corresponding equations for the extensional vibrations contain certain additional terms depending upon the square of the thickness.

2. For stresses and elastic constants, the notation of Thomson and Tait will be employed; but Mr. Love's† notation will be employed for strains and directions. The values of the various quantities involved, such as displacements, strains, and the like, at a point P on the middle surface of a plate, will be denoted by *unaccented* letters, and the values of the same quantities at a point P' on the normal at P whose distance from P is h', by *accented* letters. We shall also denote the thickness of the plate by $2h$.

The values of all the quantities with which we are concerned are

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* *Proc. Lond. Math. Soc.*, Vol. xx., p. 225.
† *Phil. Trans.*, 1888.

functions of the position of P', and are therefore of the form $F(h', x, y)$. It therefore follows that, if $\mathfrak{Q}'$ denote the value of any such quantity at P', we shall have

$$\mathfrak{Q}' = F(h', x, y)$$

$$= F + h'\left(\frac{dF}{dz}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2F}{dz^2}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the brackets denote the values of the differential coefficients at the middle surface, where $z = 0$.

Objections have been raised by Saint-Venant, and endorsed by Mr. Love, to the supposition that the displacements, strains, &c. can be expanded in powers of h' by Taylor's theorem. If, however, this objection were valid, it would follow that some of the differential coefficients, and therefore some of the quantities involved, would become infinite at some point of the middle surface, which would involve a rupture of the material, and this state of things is entirely outside the conditions of the problem.

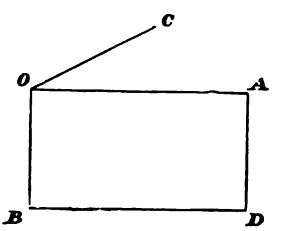
We may therefore put

$$R' = A + A_1 h' + \tfrac{1}{2}A_2 h'^2 + \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where R' is the value of R at P'; A and A_1 are at most of the orders h^3, h, respectively; and A_2 does not contain any negative powers of h.

3. We shall now consider the resultant stresses which act upon an element of a plane plate.

Let $OADB$ be a rectangle, whose sides OA, OB are parallel to the axes of x and y, described on the middle surface of the plate; and let OC be the normal at O.



The resultant stresses per unit of length, due to the action of contiguous portions of the plate, are completely specified by the following quantities; viz., on the edge AD,

T_1 = a tension across AD parallel to OA,

M_2 = a tangential shearing stress along AD,

N_2 = a normal shearing stress parallel to OC,

G_2 = a flexural couple from C to A, whose axis is parallel to AD,

H_1 = a torsional couple from B to C, whose axis is parallel to OA.

Similarly, the resultant stresses per unit of length, which act on the edge BD, are

T_1 = a tension across BD parallel to OB,

M_1 = a tangential shearing stress along BD,

N_1 = a normal shearing stress parallel to OC,

G_1 = a flexural couple from B to C, whose axis is parallel to BD,

H_1 = a torsional couple from C to A, whose axis is parallel to OB.

If the edges AD or BD were of finite length, there would also be a couple about an axis parallel to OC; but, since this couple is proportional to the cube of the edge, it vanishes in comparison with the other stresses when the rectangle $OADB$ is indefinitely diminished.

4. The six equations connecting the strains and displacements are

$$\left.\begin{aligned} \sigma_1' = \frac{du'}{dx}, \quad \sigma_2' = \frac{dv'}{dy}, \quad \sigma_3' = \frac{dw'}{dz}, \\ \varpi_1' = \frac{dv'}{dz} + \frac{dw'}{dy}, \quad \varpi_2' = \frac{dw'}{dx} + \frac{du'}{dz}, \quad \varpi_3' = \frac{du'}{dy} + \frac{dv'}{dx} \end{aligned}\right\} \quad \ldots\ldots(3).$$

Also,
$$\begin{aligned} R' &= (m+n)\,\sigma_3' + (m-n)(\sigma_1' + \sigma_2') \\ &= (m+n)\,\sigma_3 + (m-n)(\sigma_1 + \sigma_2) \\ &\quad + \left\{ (m+n)\left(\frac{d\sigma_3}{dz}\right) + (m-n)(\lambda+\mu) \right\} h' + \ldots, \end{aligned}$$

where λ, μ are written for $(d\sigma_1/dz)$, $(d\sigma_2/dz)$, respectively. Comparing this with (2), we see that

$$(m+n)\,\sigma_3 + (m-n)(\sigma_1 + \sigma_2) = A \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$(m+n)\left(\frac{d\sigma_3}{dz}\right) + (m-n)(\lambda+\mu) = A_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

We can now calculate the values of the couples. We have

$$G_1 = -\int_{-h}^{h} Q' h' dh' = -\tfrac{2}{3}h^3 \left(\frac{dQ}{dz}\right),$$

and
$$\left(\frac{dQ}{dz}\right) = (m+n)\,\mu + (m-n)\left\{\lambda + \left(\frac{d\sigma_3}{dz}\right)\right\};$$

whence, putting
$$E = \frac{m-n}{m+n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

we obtain
$$G_1 = -\tfrac{4}{3}nh^3 \left\{\mu + E\,(\lambda+\mu) + EA_1/2n\right\} \quad \ldots\ldots\ldots(6\text{A}).$$

Similarly, $$G_2 = \tfrac{4}{3}nh^3\left\{\lambda + E(\lambda+\mu) + EA_1/2n\right\} \quad \text{.....(6B).}$$

But, since A_1 is at most of the order h, it follows that this quantity may be neglected when multiplied by h^3; we thus obtain

$$G_1 = -\tfrac{4}{3}nh^3\left\{\mu + E(\lambda+\mu)\right\} \quad \text{.....(7),}$$

$$G_2 = \tfrac{4}{3}nh^3\left\{\lambda + E(\lambda+\mu)\right\} \quad \text{.....(8).}$$

Also, $$H_1 = -H_2 = -\tfrac{2}{3}nh^3\left(\frac{d\varpi_3}{dz}\right) \quad \text{.....(9),}$$

$$N_1 = \int_{-h}^{h} \varpi_1' dh', \quad N_2 = \int_{-h}^{h} \varpi_2' dh' \quad \text{.....(10).}$$

5. We must now obtain the equations of motion.

Let X, Y, Z be the bodily forces per unit of mass, which act on the plate; then, taking moments about OC, it follows at once that $M_1 = M_2$; also, resolving parallel to OA, OB, OC, and taking moments about OA, OB, we obtain the following five equations, viz.:

$$\left.\begin{aligned}
\frac{dT_1}{dx} + \frac{dM}{dy} &= \rho\int_{-h}^{h} \ddot{u}'dh' - 2\rho hX \\
\frac{dT_2}{dy} + \frac{dM}{dx} &= \rho\int_{-h}^{h} \ddot{v}'dh' - 2\rho hY \\
\frac{dN_2}{dx} + \frac{dN_1}{dy} &= \rho\int_{-h}^{h} \ddot{w}'dh' - 2\rho hZ \\
\frac{dG_1}{dy} + \frac{dH_1}{dx} + N_1 &= -\rho\int_{-h}^{h} \ddot{v}'h'dh' \\
\frac{dG_2}{dx} + \frac{dH_2}{dy} - N_2 &= \rho\int_{-h}^{h} \ddot{u}'h'dh'
\end{aligned}\right\} \quad \text{.....(11).}$$

Expanding in powers of h', and using the fourth of (3), we obtain

$$-\int_{-h}^{h} \ddot{v}'h'dh' = \tfrac{2}{3}h^3\left(\frac{d\ddot{w}}{dy} - \ddot{\varpi}_1\right).$$

Now, in whatever manner ϖ_1 may depend upon h, it evidently cannot contain any negative powers of h, otherwise ϖ_1 would increase indefinitely as the thickness of the shell diminished indefinitely; we therefore see that the right-hand side of the fourth of (11) is of the order h^3, and, since G_1, H_1 are of the order h^3, it follows that N_1 is also of the order h^3; a similar argument applies to N_2, whence, by (10), ϖ_1', ϖ_2' must each be of the order of the square of the thickness, and therefore produce terms of the order h^5 in the expression for the potential energy, which are to be omitted.

We can now calculate the right-hand sides of (11); for, putting

$$K = \sigma_1 + \sigma_2,$$

we have $$\int_{-h}^{h} \ddot{u}'dh' = \int_{-h}^{h} \left\{ \ddot{u} + h' \left(\frac{d\ddot{u}}{dx}\right) + \tfrac{1}{2}h'^2 \left(\frac{d^2\ddot{u}}{dx^2}\right) \right\} dh'$$

$$= 2h\ddot{u} + \tfrac{1}{3}h^3 \left(\frac{d^2\ddot{u}}{dx^2}\right).$$

But, from equations (3),

$$\frac{d^2u'}{dz^2} = \frac{d\varpi_1'}{dz} - \frac{d\sigma_3'}{dx}$$

$$= \frac{d\varpi_1'}{dz} - \frac{1}{m+n}\frac{dK'}{dx} + E\frac{dK}{dx}.$$

Since the first two terms on the right-hand side are of the order h^3, they may be neglected when multiplied by h^3; whence

$$\int_{-h}^{h} \ddot{u}'dh' = 2h\ddot{u} + \tfrac{1}{3}h^3 E \frac{d\ddot{K}}{dx}.$$

Similarly,
$$\int_{-h}^{h} \ddot{v}'dh' = 2h\ddot{v} + \tfrac{1}{3}h^3 E \frac{d\ddot{K}}{dy},$$

$$\int_{-h}^{h} \ddot{w}'dh' = 2h\ddot{w} + \tfrac{1}{3}h^3 E\nabla^2\ddot{w},$$

and our equations become

$$\frac{dT_1}{dx} + \frac{dM}{dy} = 2\rho h(\ddot{u} - X) + \tfrac{1}{3}\rho h^3 E \frac{d^2\ddot{K}}{dx} \quad \text{.....(12),}$$

$$\frac{dT_2}{dy} + \frac{dM}{dx} = 2\rho h(\ddot{v} - Y) + \tfrac{1}{3}\rho h^3 E \frac{d^2\ddot{K}}{dy} \quad \text{.....(13),}$$

$$\frac{dN_2}{dx} + \frac{dN_1}{dy} = 2\rho h(\ddot{w} - Z) + \tfrac{1}{3}\rho h^3 E\nabla^2\ddot{w} \quad \text{.....(14),}$$

$$\frac{dG_1}{dy} + \frac{dH_1}{dx} + N_1 = \tfrac{2}{3}\rho h^3 \frac{d\ddot{w}}{dy} \quad \text{.....(15),}$$

$$\frac{dG_2}{dx} + \frac{dH_2}{dy} - N_2 = -\tfrac{2}{3}\rho h^3 \frac{d\ddot{w}}{dx} \quad \text{.....(16).}$$

Again,
$$\left.\begin{aligned} \lambda &= \frac{d\sigma_1}{dz} = \frac{d^2u}{dz\,dx} = -\frac{d^2w}{dx^2} \\ \mu &= -\frac{d^2w}{dy^2}, \quad \left(\frac{d\varpi_3}{dz}\right) = -2\frac{d^2w}{dx\,dy} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(17).$$

whence
$$\left.\begin{aligned} G_1 &= \tfrac{4}{3}nh^3\left(\frac{d^2w}{dy^2} + E\nabla^2 w\right) \\ G_2 &= -\tfrac{4}{3}nh^3\left(\frac{d^2w}{dx^2} + E\nabla^2 w\right) \\ H_1 &= -H_2 = \tfrac{4}{3}nh^3\frac{d^2w}{dx\,dy} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

We can now obtain the ordinary equation for determining the flexural vibrations of the plate. Eliminating N_1, N_2 from (14), by means of (15) and (16), and then substituting the values of G_1, G_2, H from (18), we obtain

$$\tfrac{4}{3}nh^3(1+E)\nabla^4 w + 2\rho(\ddot{w} - Z) + \tfrac{1}{3}\rho h^2(E-2)\nabla^2\ddot{w} = 0 \ldots\ldots(19).$$

If $Z = 0$, the last term must be omitted, since it gives rise to a term h^4 in the period, whence (19) becomes

$$\frac{4mnh^2}{3(m+n)}\nabla^4 w + \rho\ddot{w} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

6. Equation (20) and the values of the couples have been obtained without having made any use of the expression for the potential energy, or having employed the Calculus of Variations. We shall now complete the solution for the flexural vibrations by obtaining the boundary conditions. It will be hereafter proved, as is almost obvious, that T_1, T_2, and M depend solely upon the extension of the middle surface, and are therefore zero when the vibrations are purely flexural. We are not, therefore, now concerned with those quantities. Now one of the boundary conditions at a free edge requires that the flexural couple about the edge should vanish; hence, if the free edge is parallel to the axis of y, $G_2 = 0$, and therefore, by the second of (18), one boundary condition is

$$\frac{d^2w}{dx^2} + E\nabla^2 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

This is the interpretation of the second of equations (6) given by Lord Rayleigh ("Theory of Sound," Vol. I., p. 299).

With regard to the other boundary conditions, it was supposed by

Poisson that it was necessary that the normal shearing stress and the torsional couple at a free edge should vanish, and thus there would be two more equations of condition. Kirchhoff has, however, shown that only one additional equation is necessary, and the reason of this is that it is possible to apply a certain distribution of stress to the edge of a plate without producing any alteration in the potential energy.

A proof of this proposition, founded on general reasoning, is given in Thomson and Tait, § 646; but the theorem may be established analytically as follows.

By Stokes' theorem,

$$\int\left(\frac{dH'}{dx}\delta w+H'\frac{d\delta w}{dx}\right)dx+\int\left(\frac{dH'}{dy}\delta w+H'\frac{d\delta w}{dy}\right)dy=0,$$

the integration extending once round the rectangle *OADB*.

If we apply to the side *AD* the stresses

$$N_2'=\frac{dH'}{dy},\quad H_1'=H';$$

to the side *DB* the stresses

$$N_1'=\frac{dH'}{dx},\quad H_2'=-H';$$

and to the sides *BO*, *OA* corresponding and opposite stresses, the preceding integral becomes

$$\int\left(N_1'\delta w-H_2'\frac{d\delta w}{dx}\right)dx+\int\left(N_2'\delta w+H_1'\frac{d\delta w}{dy}\right)dy=0,$$

which shows that the work done by this system of stresses is zero. Such a system of stresses may therefore be applied or removed without interfering with the equilibrium or motion of the plate.

If, therefore, we suppose that the rectangle *OADB*, instead of being under the action of the remainder of the plate, is isolated, and that constraining stresses are applied to its edges, then it follows that if, instead of the torsional couples H_1, $-H_1$ (since $H_2=-H_1$), due to the action of contiguous portions of the plate, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1=H_1+H' \quad \text{.....(22),}$$

$$\mathfrak{H}_2=-H_1-H' \quad \text{.....(23),}$$

the energy and state of strain will be unaltered, provided we apply, in addition, the stresses

$$\mathfrak{N}_2 = N_2 + \frac{dH'}{dy} \quad \ldots\ldots(24),$$

$$\mathfrak{N}_1 = N_1 + \frac{dH'}{dx} \quad \ldots\ldots(25).$$

Whence, eliminating H' between (22) and (24), and between (23) and (25), respectively, we obtain

$$\left.\begin{aligned} \mathfrak{N}_2 - \frac{d\mathfrak{N}_1}{dy} &= N_2 - \frac{dH_1}{dy} \\ \mathfrak{N}_1 + \frac{d\mathfrak{N}_2}{dx} &= N_1 + \frac{dH_2}{dx} \end{aligned}\right\} \quad \ldots\ldots(26).$$

In these equations the Roman letters denote the values of the stresses due to the action of contiguous portions of the plate, whilst the Old English letters denote the values of the actual stresses applied to the boundary; whence the conditions to be satisfied at a free edge are

$$N_2 - \frac{dH_1}{dy} = 0 \quad \ldots\ldots(27)$$

along an edge parallel to the axis of y, and

$$N_1 - \frac{dH_1}{dx} = 0 \quad \ldots\ldots(28)$$

along an edge parallel to the axis of x.

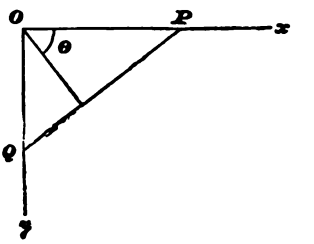
Since the period of the flexural vibrations depends upon the square of the thickness, the terms on the right-hand sides of (15) and (16) may be omitted; these equations, combined with (18), thus determine N_1, N_2, and (27) becomes

$$\frac{d}{dx}\left\{\frac{d^2w}{dx^2} + \frac{3m-n}{2m}\frac{d^2w}{dy^2}\right\} = 0 \quad \ldots\ldots(29),$$

which is the first of equations (6) of Lord Rayleigh's "Theory of Sound," above referred to.

[6*. It has been suggested to me that it would be desirable to apply the foregoing method to obtain the general boundary conditions when the free edge is any curve. In order to do this, let PQ be an

element of the curved boundary, and let θ be the angle which the normal to the element makes with the axis of x; let $OP = \delta x$, $OQ = \delta y$, $PQ = \delta s$, and let the stresses across PQ be denoted by accented letters.



Resolving parallel and perpendicularly to PQ, and along the normal, we at once obtain the following equations, viz.:

$$\left.\begin{aligned} T' &= T_1 \cos^2\theta + T_2 \sin^2\theta + M \sin 2\theta, \\ M' &= \tfrac{1}{2}(T_2 - T_1) \sin 2\theta + M \cos 2\theta, \\ N' &= N_1 \sin\theta + N_2 \cos\theta, \\ G' &= G_2 \cos^2\theta - G_1 \sin^2\theta - H_1 \sin 2\theta, \\ H' &= \tfrac{1}{2}(G_1 + G_2) \sin 2\theta + H_1 \cos 2\theta, \end{aligned}\right\} \quad \ldots\ldots\ldots (30).$$

Substituting the values of G_1, G_2, H_1, from (18), in the fourth of (30), we obtain

$$G' = -\tfrac{4}{3}nh^3 \left\{ \frac{d^2w}{dx^2}\cos^2\theta + \frac{d^2w}{dy^2}\sin^2\theta + E\nabla^2 w + \frac{d^2w}{dx\,dy}\sin 2\theta \right\} \quad \ldots (31).$$

At a free edge $G' = 0$, and therefore the right hand of (31) must vanish. This is the interpretation of the second boundary condition given by Lord Rayleigh—"Theory of Sound," Vol. I., p. 297, equations (9).

From (18) and the last of (30), we obtain

$$H' = \tfrac{4}{3}nh^3 \left\{ \left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2} \right) \sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta) \frac{d^2w}{dx\,dy} \right\}.$$

Now, if ξ be any single valued function,

$$\int \left(\frac{d\xi}{ds}\delta w + \xi \frac{d\delta w}{ds} \right) ds = 0,$$

the integration extending once round the curved boundary.

The work done by the stresses N' and H', due to the action of contiguous portions of the plate, is

$$\int \left(N'\delta w + H' \frac{d\delta w}{ds} \right) ds.$$

If, therefore, we apply to the curved boundary the stresses $\mathfrak{P}'$, $\mathfrak{R}'$, where

$$\mathfrak{R}' = H' + \xi,$$

$$\mathfrak{P}' = N' + \frac{d\xi}{ds},$$

the energy and state of strain will be unaltered; whence, eliminating ξ, we obtain

$$\mathfrak{P}' - \frac{d\mathfrak{R}'}{ds} = N' - \frac{dH'}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32).$$

At a free edge $\mathfrak{P}'$, $\mathfrak{R}'$ are zero, and therefore the other boundary condition is that the right-hand side of (28c) should vanish.

When there are no impressed forces, $\ddot{w} \propto h^3$, and therefore the right-hand sides of (15) and (16) may be omitted; whence, by (15), (16), and the third of (30),

$$N' = -\left(\frac{dG_1}{dy} + \frac{dH_1}{dx}\right)\sin\theta + \left(\frac{dG_2}{dx} - \frac{dH_1}{dy}\right)\cos\theta$$

$$= -\frac{dG_1}{dy}\sin\theta + \frac{dG_2}{dx}\cos\theta - \frac{dH_1}{ds}.$$

But

$$G_1 = \tfrac{4}{3}nh^3\left\{(1+E)\nabla^2 w - \frac{d^2w}{dx^2}\right\},$$

$$G_2 = -\tfrac{4}{3}nh^3\left\{(1+E)\nabla^2 w - \frac{d^2w}{dy^2}\right\},$$

whence, if dn be an element of the normal to the bounding curve drawn outwards,

$$N' = -\tfrac{4}{3}nh^3\left\{(1+E)\frac{d}{dn}\nabla^2 w - \frac{d^3w}{ds\,dx\,dy}\right\} - \frac{dH_1}{ds}$$

$$= -\tfrac{4}{3}nh^3(1+E)\frac{d}{dn}\nabla^2 w.$$

The second boundary condition at a free edge, therefore, becomes

$$(1+E)\frac{d}{dn}\nabla^2 w + \frac{d}{ds}\left\{\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right)\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\frac{d^2w}{dx\,dy}\right\} = 0,$$

which is the first of equations (9) given on p. 297 of Lord Rayleigh's "Theory of Sound." *Added Feb. 19th*, 1890.]

7. Having discussed the flexural vibrations of a plane plate, we shall now investigate the expression for the potential energy.

By the ordinary formula, the potential energy of the element $2h\,dS$ is

$$W = \int_{-h}^{h} \left[(m+n)\,\Delta'^2 + n\left\{ \varpi_1'^2 + \varpi_2'^2 + \varpi_3'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1') \right\} \right] dh' \quad \ldots\ldots\ldots\ldots (33),$$

where Δ' is the dilatation.

Expanding in powers of h', by Taylor's theorem, we have

$$\Delta' = \Delta + \left(\frac{d\Delta}{dz}\right) h' + \tfrac{1}{2}\left(\frac{d^2\Delta}{dz^2}\right) h'^2 + \ldots,$$

whence

$$\tfrac{1}{2}(m+n) \int_{-h}^{h} \Delta'^2 dh' = (m+n) \left\{ h\Delta^2 + \tfrac{1}{3}h^3 \left(\frac{d\Delta}{dz}\right)^2 + \tfrac{1}{3}h^3 \left(\frac{d^2\Delta}{dz^2}\right)\Delta + \ldots \right\},$$

from which it is easily seen that W is expressible in a series of odd powers of h.

Let us now write λ, μ, p; λ', μ', p' for the values of the first and second differential coefficients of σ_1, σ_2, ϖ_3, with respect to z, at the middle surface of the shell; so that $\lambda = (d\sigma_1/dz)$, &c.

In whatever way R' may depend upon h, we have shown that it may be expressed in the form

$$R' = A + A_1 h' + \tfrac{1}{2} A_2 h'^2 + \ldots,$$

where the A's are quantities which depend upon the thickness, and also upon quantities which define the state of strain of the middle surface But

$$\begin{aligned} R' &= (m+n)\,\sigma_3' + (m-n)\,(\sigma_1' + \sigma_2') \\ &= (m+n)\,\sigma_3' + (m-n)\,(\sigma_1 + \sigma_2) \\ &\quad + \left\{ (m+n)\left(\frac{d\sigma_3}{dz}\right) + (m-n)(\lambda+\mu) \right\} h' \\ &\quad + \tfrac{1}{2}\left\{ (m+n)\left(\frac{d^2\sigma_3}{dz^2}\right) + (m-n)(\lambda'+\mu') \right\} h'^2 + \&c.; \end{aligned}$$

whence

$$(m+n)\,\sigma_3 + (m-n)\,(\sigma_1+\sigma_2) = A,$$

$$(m+n)\left(\frac{d\sigma_3}{dz}\right) + (m-n)\,(\lambda+\mu) = A_1,$$

$$(m+n)\left(\frac{d^2\sigma_3}{dz^2}\right) + (m-n)\,(\lambda'+\mu') = A_2.$$

From these equations, we obtain

$$\Delta = (1-E)(\sigma_1+\sigma_2)+\frac{A}{m+n},$$

$$\left(\frac{d\Delta}{dz}\right) = (1-E)(\lambda+\mu)+\frac{A_1}{m+n},$$

$$\left(\frac{d^2\Delta}{dz^2}\right) = (1-E)(\lambda'+\mu')+\frac{A_2}{m+n},$$

and therefore the portion of W depending on Δ' is

$$\frac{4n^2}{m+n}\left\{h(\sigma_1+\sigma_2+A/2n)^2+\tfrac{1}{3}h^3(\lambda+\mu+A_1/2n)^2 + \tfrac{1}{3}h^3(\sigma_1+\sigma_2+A/2n)(\lambda'+\mu'+A_2/2n)\right\} \ldots (34).$$

Again,

$$\sigma_1'\sigma_2' = \sigma_1\sigma_2+\lambda\mu h'^2+\tfrac{1}{2}(\lambda'\sigma_2+\mu'\sigma_1)h'^2+\ldots;$$

whence

$$2n\int_{-h}^{h}\sigma_1'\sigma_2'dh' = 4nh\sigma_1\sigma_2+\tfrac{4}{3}nh^3\lambda\mu+\tfrac{2}{3}nh^3(\lambda'\sigma_2+\mu'\sigma_1) \ldots\ldots (35).$$

Also,

$$(\sigma_1'+\sigma_2')\sigma_2' = (\sigma_1+\sigma_2)\sigma_2+h'^2(\lambda+\mu)\left(\frac{d\sigma_2}{dz}\right) + \tfrac{1}{2}h'^2(\lambda'+\mu')\sigma_2+\tfrac{1}{2}h'^2(\sigma_1+\sigma_2)\left(\frac{d^2\sigma_2}{dz^2}\right);$$

whence

$$2n\int_{-h}^{h}(\sigma_1'+\sigma_2')\sigma_2'dh' = 4nh(\sigma_1+\sigma_2)\sigma_2+\tfrac{4}{3}nh^3(\lambda+\mu)\left(\frac{d\sigma_2}{dz}\right) + \tfrac{2}{3}nh^3(\lambda'+\mu')\sigma_2+\tfrac{4}{3}nh^3(\sigma_1+\sigma_2)\left(\frac{d^2\sigma_2}{dz^2}\right)$$

$$= 4nh(\sigma_1+\sigma_2)\left\{\frac{A}{m+n}-E(\sigma_1+\sigma_2)\right\} + \tfrac{4}{3}nh^3(\lambda+\mu)\left\{\frac{A_1}{m+n}-E(\lambda+\mu)\right\}$$

$$+\tfrac{2}{3}nh^3\left\{\frac{A(\lambda'+\mu')+A_2(\sigma_1+\sigma_2)}{m+n}-2E(\sigma_1+\sigma_2)(\lambda'+\mu')\right\} \ldots\ldots (36).$$

Finally,

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi_2'^2dh' = n(h\varpi_2^2+\tfrac{1}{3}h^3p^2+\tfrac{1}{3}h^3\varpi_2p') \ldots\ldots\ldots (37),$$

and therefore the value of W, so far as it depends upon σ_1', σ_2' σ_3', ϖ_3', is

$$W = 2nh\left\{\sigma_1^2+\sigma_2^2+E(\sigma_1+\sigma_2)^2+\tfrac{1}{2}\varpi_3^2+\frac{A^2}{2n(m+n)}\right\}$$

$$+\tfrac{2}{3}nh^3\left\{\lambda^2+\mu^2+E(\lambda+\mu)^2+\tfrac{1}{2}p^2+\frac{A_1^2}{2n(m+n)}\right\}$$

$$+\tfrac{2}{3}nh^3\left[\lambda'\left\{\sigma_1+E(\sigma_1+\sigma_2)\right\}+\mu'\left\{\sigma_2+E(\sigma_1+\sigma_2)\right\}+\tfrac{1}{2}\varpi_3 p'+\frac{AA_2}{2n(m+n)}\right]$$

.....(38).

Since ϖ_1, ϖ_2 are quantities of the order of the square of the thickness, the portion of the potential energy which depends upon them is of the order h^5, and must therefore be omitted; also, since R' is of the order h^3, the portion of W which depends upon A, A_1, A_2 is of a higher order than h^3, and must also be omitted. The correct expression for the potential energy to the order h^3 is therefore obtained by putting A, A_1, A_2 zero in (38). From (3) and (17), we see that the first line depends solely upon the extension of the middle surface, and therefore upon u and v, and not upon w; the second line depends solely upon w, and not upon u and v, and is the ordinary expression for the potential energy due to bending. As regards the third line, we have

$$\lambda' = \left(\frac{d^2\sigma_1}{dz^2}\right) = \left(\frac{d^3u}{dx\,dz^2}\right) = -\left(\frac{d^3w}{dx^2dz}\right).$$

If, therefore, we put $K = \sigma_1+\sigma_2$,

we obtain $\lambda' = E\dfrac{d^2K}{dx^2}$, $\mu' = E\dfrac{d^2K}{dy^2}$, $p' = 2E\dfrac{d^2K}{dx\,dy}$ .....(39),

from which it follows that the last line of (38) depends solely upon u and v.

9. In order to obtain the correct values of T_1, T_2, M as far as h^3, and the equations which the u and v vibrations satisfy, we shall proceed to form the variational equation; but in so doing we may omit all the terms in w, since by retaining them we should merely reproduce the results which we have already obtained and discussed. On this understanding, let us apply the Principle of Virtual Work to a rectangle bounded by four lines parallel to the axes; then, if δW denote the variation of W, and $\delta\mathfrak{T}$ the portion of the variational equation due to the accelerations, δV the work done by the bodily

forces, and $\delta\mathfrak{T}$ the work done upon the rectangle by the stresses arising from the action of contiguous portions of the plate; the variational equation is

$$\delta W + \delta\mathfrak{U} = \delta V + \delta\mathfrak{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40).$$

We must now calculate $\delta\mathfrak{U}$.

$$\delta\mathfrak{U} = \rho \iiint_{-h}^{h} (\ddot{u}'\delta u' + \ddot{v}'\delta v' + \ddot{w}'\delta w')\, dh'\, dx\, dy,$$

and $$\int_{-h}^{h} \ddot{u}'\delta u'\, dh' = 2h\ddot{u}\delta u + \tfrac{2}{3}h^3 \frac{d\ddot{w}}{dx}\frac{d\delta w}{dx} + \tfrac{1}{3}h^3 E\left(\ddot{u}\frac{d\delta K}{dx} + \frac{d\ddot{K}}{dx}\delta u\right),$$

$$\int_{-h}^{h} \ddot{v}'\delta v'\, dh' = 2h\ddot{v}\delta v + \tfrac{2}{3}h^3 \frac{d\ddot{w}}{dy}\frac{d\delta w}{dy} + \tfrac{1}{3}h^3 E\left(\ddot{v}\frac{d\delta K}{dy} + \frac{d\ddot{K}}{dy}\delta v\right),$$

$$\int_{-h}^{h} \ddot{w}'\delta w'\, dh' = 2h\ddot{w}\delta w + \tfrac{2}{3}h^3 E^2 \ddot{K}\delta K + \tfrac{1}{3}h^3 E\,(\ddot{w}\nabla^2\delta w + \nabla^2\ddot{w}\delta w);$$

whence, omitting terms in w,

$$\delta\mathfrak{U} = 2\rho h \iint (\ddot{u}\delta u + \ddot{v}\delta v)\, dx\, dy + \tfrac{2}{3}\rho h^3 E^2 \iint \ddot{K}\delta K\, dx\, dy$$

$$+ \tfrac{1}{3}\rho h^3 E \iint \left(\ddot{u}\frac{dK}{dx} + \ddot{v}\frac{d\delta K}{dy} + \frac{d\ddot{K}}{dx}\delta u + \frac{d\ddot{K}}{dy}\delta v\right) dx\, dy \ldots\ldots (41).$$

To find δV, we have

$$\delta V = \rho \iiint_{-h}^{h} (X\delta u' + Y\delta v' + Z\delta w')\, dh'\, dx\, dy$$

$$= 2h\rho \iint (X\delta u + Y\delta v + Z\delta w)\, dx\, dy$$

$$+ \tfrac{1}{3}\rho h^3 \iint \left\{ X\left(\frac{d^2\delta u}{dx^2}\right) + Y\left(\frac{d^2\delta v}{dx^2}\right) + Z\left(\frac{d^2\delta w}{dx^2}\right)\right\} dx\, dy.$$

But $$\left(\frac{d^2u}{dx^2}\right) = E\frac{dK}{dx}, \quad \left(\frac{d^2v}{dx^2}\right) = E\frac{dK}{dy}, \quad \left(\frac{d^2w}{dx^2}\right) = E\nabla^2 w;$$

whence the portion of δV which depends upon u and v is

$$\delta V = 2h\rho \iint (X\delta u + Y\delta v)\, dx\, dy + \tfrac{1}{3}\rho E h^3 \iint \left(X\frac{d\delta K}{dx} + Y\frac{d\delta K}{dy}\right) dx\, dy$$

$$\ldots\ldots\ldots\ldots (42).$$

In order to find $\delta\mathfrak{L}$, let

$$\mathfrak{A} = \sigma_1 + E(\sigma_1+\sigma_2), \quad \mathfrak{B} = \sigma_2 + E(\sigma_1+\sigma_2) \quad \ldots\ldots\ldots(43);$$

then, writing ϖ for ϖ_3, we have

$$P = 2n\mathfrak{A}, \quad Q = 2n\mathfrak{B}, \quad U = n\varpi \ldots\ldots\ldots\ldots\ldots(44),$$

and

$$\int_{-h}^{h} P'\delta u'dh' = \left\{2hP + \tfrac{1}{3}h^3\left(\frac{d^2P}{dx^2}\right)\right\}\delta u + \tfrac{2}{3}h^3\left(\frac{dP}{dx}\right)\left(\frac{d\delta u}{dx}\right) + \tfrac{1}{3}h^3P\left(\frac{d^2\delta u}{dx^2}\right)$$

$$= T_1\delta u - G_2\frac{d\delta w}{dx} + \tfrac{2}{3}nh^3E\mathfrak{A}\frac{d\delta K}{dx},$$

$$\int_{-h}^{h} U'\delta v'dh' = M\delta v + H_1\frac{d\delta w}{dy} + \tfrac{1}{3}nh^3E\varpi\frac{d\delta K}{dy},$$

$$\int_{-h}^{h} Q'\delta v'dh' = T_2\delta v + G_1\frac{d\delta w}{dy} + \tfrac{2}{3}nh^3E\mathfrak{B}\frac{d\delta K}{dy},$$

$$\int_{-h}^{h} U'\delta u'dh' = M\delta u - H_2\frac{d\delta w}{dx} + \tfrac{1}{3}nh^3E\varpi\frac{d\delta K}{dx};$$

whence, omitting terms in w,

$$\delta\mathfrak{L} = \int\left(T_1\delta u + M\delta v + \tfrac{2}{3}nh^3E\mathfrak{A}\frac{d\delta K}{dx} + \tfrac{1}{3}nh^3E\varpi\frac{d\delta K}{dy}\right)dy$$

$$+ \int\left(M\delta u + T_2\delta v + \tfrac{2}{3}nh^3E\mathfrak{B}\frac{d\delta K}{dy} + \tfrac{1}{3}nh^3E\varpi\frac{d\delta K}{dx}\right)dx \quad \ldots\ldots(45).$$

By means of (34), (39), (43), the values of the first and third lines of W may be written in the form

$$W = 2nh\iint\left\{\sigma_1^2 + \sigma_2 + E(\sigma_1+\sigma_2)^2 + \tfrac{1}{2}\varpi^2\right\}dxdy$$

$$+ \tfrac{2}{3}nh^3E\iint\left(\mathfrak{A}\frac{d^2K}{dx^2} + \mathfrak{B}\frac{d^2K}{dy^2} + \varpi\frac{d^2K}{dxdy}\right)dxdy \quad \ldots\ldots(46),$$

which we shall denote by W_1, W_2; whence

$$\delta W_1 = 4nh\int(\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\delta v)\,dy + 4nh\int(\mathfrak{B}\delta v + \tfrac{1}{2}\varpi\delta u)\,dx$$

$$- 4nh\iint\left\{\left(\frac{d\mathfrak{A}}{dx} + \frac{1}{2}\frac{d\varpi}{dy}\right)\delta u + \left(\frac{d\mathfrak{B}}{dy} + \frac{1}{2}\frac{d\varpi}{dx}\right)\delta v\right\}dxdy \ldots(47).$$

Since δW_2 depends upon h^3, we obtain, from (40), (41), (43), (45),

and (47), the approximate equations

$$\left.\begin{aligned} 2n\left(\frac{d\mathfrak{A}}{dx}+\frac{1}{2}\frac{d\varpi}{dy}\right)&=\rho\,(\ddot{u}-X),\\ 2n\left(\frac{d\mathfrak{B}}{dy}+\frac{1}{2}\frac{d\varpi}{dx}\right)&=\rho\,(\ddot{v}-Y),\\ T_1=4nh\mathfrak{A},\quad T_2=4nh\mathfrak{B},\quad M&=2nh\varpi, \end{aligned}\right\}\ \ldots\ldots\ldots\ldots(48);$$

from which we see that

$$\rho\left(\ddot{K}-\frac{dX}{dx}-\frac{dY}{dy}\right)=2n\left(\frac{d^2\mathfrak{A}}{dx^2}+\frac{d^2\mathfrak{B}}{dy^2}+\frac{d^2\varpi}{dx\,dy}\right)=2n\,(1+E)\,\nabla^2K \quad\ldots\ldots\ldots\ldots(49).$$

In these equations the terms in h^3 are omitted. Again,

$$\delta W_2=\tfrac{2}{3}nh^3E\iint\left(\frac{d^2K}{dx^2}\delta\mathfrak{A}+\frac{d^2K}{dy^2}\delta\mathfrak{B}+\frac{d^2K}{dx\,dy}\delta\varpi\right.$$
$$\left.+\mathfrak{A}\frac{d^2\delta K}{dx^2}+\mathfrak{B}\frac{d^2\delta K}{dy^2}+\varpi\frac{d^2\delta K}{dx\,dy}\right)dx\,dy \quad\ldots\ldots(50).$$

By a single integration by parts, the value of the last three terms of (50) is

$$\tfrac{2}{3}nh^3E\int\left(\mathfrak{A}\frac{d\delta K}{dx}+\tfrac{1}{2}\varpi\frac{d\delta K}{dy}\right)dy+\tfrac{2}{3}nh^3E\int\left(\mathfrak{B}\frac{d\delta K}{dy}+\tfrac{1}{2}\varpi\frac{d\delta K}{dx}\right)dx$$
$$-\tfrac{2}{3}nh^3E\iint\left\{\left(\frac{d\mathfrak{A}}{dx}+\frac{1}{2}\frac{d\varpi}{dy}\right)\frac{d\delta K}{dx}+\left(\frac{d\mathfrak{B}}{dy}+\frac{1}{2}\frac{d\varpi}{dx}\right)\frac{d\delta K}{dy}\right\}dx\,dy$$
$$\ldots\ldots\ldots\ldots(51).$$

Since the right-hand side of (50) is multiplied by h^3, we may substitute for the coefficients of $d\delta K/dx$, &c., their approximate values from (48). Doing this, and substituting the value of δW_2, and also the values of $\delta\mathfrak{T}$, δV, $\delta\mathfrak{L}$, from (41), (42), and (45), in (40), it will be found that several of the terms cancel one another, and (40) becomes

$$\delta W_1+\tfrac{2}{3}nh^3E\iint\left(\frac{d^2K}{dx^2}\delta\mathfrak{A}+\frac{d^2K}{dy^2}\delta\mathfrak{B}+\frac{d^2K}{dx\,dy}\delta\varpi\right)dx\,dy$$
$$+2\rho h\iint(\ddot{u}\delta u+\ddot{v}\delta v)\,dx\,dy+\tfrac{2}{3}\rho h^3E^2\iint\ddot{K}\delta K\,dx\,dy$$
$$+\tfrac{1}{3}\rho h^3E\iint\left(\frac{d\ddot{K}}{dx}\delta u+\frac{d\ddot{K}}{dy}\delta v\right)dx\,dy$$
$$=2\rho h\iint(X\delta u+Y\delta v)\,dx\,dy+\int(T_1\delta u+M\delta v)\,dy+\int(M\delta u+T_2\delta v)\,dx$$
$$\ldots\ldots\ldots\ldots(52).$$

Integrating by parts, and substituting the approximate value of $\ddot{K}$ from (49), which we may do, since the terms in $\ddot{K}$ are all multiplied by h^3, and putting, for brevity,

$$\Omega = dX/dx + dY/dy,$$

52) becomes

$$\delta W_1 - \tfrac{4}{3}nh^3E^2(1+E)\iint\left(\frac{d\nabla^2K}{dx}\delta u + \frac{d\nabla^2K}{dy}\delta v\right)dx\,dy$$

$$+\tfrac{1}{3}\rho h^3E(1-2E)\iint\left(\frac{d\Omega}{dx}\delta u + \frac{d\Omega}{dy}\delta v\right)dx\,dy + 2\rho h\iint(\ddot{u}\delta u + \ddot{v}\delta v)\,dx\,dy$$

$$-\tfrac{2}{3}nh^3E\int\left[\left\{\frac{d^2K}{dx^2} + E(2E+3)\nabla^2K\right\}\delta u + \frac{d^2K}{dx\,dy}\delta v\right]dy + \tfrac{2}{3}\rho h^3E^2\int\Omega\delta u\,dy$$

$$+\tfrac{2}{3}nh^3E\int\left[\left\{\frac{d^2K}{dy^2} + E(2E+3)\nabla^2K\right\}\delta v + \frac{d^2K}{dx\,dy}\delta u\right]dx + \tfrac{2}{3}\rho h^3E^2\int\Omega\delta v\,dx$$

$$= 2\rho h\iint(X\delta u + Y\delta v)\,dx\,dy + \int(T_1\delta u + M\delta v)\,dy + \int(M\delta u + T_2\delta v)\,dx.$$

Substituting the value of δW_1 from (47), and equating coefficients δu, δv in the single and double integrals, we obtain the equations of motion, viz.:

$$\left.\begin{aligned}
\rho(\ddot{u}-X) &= 2n\left(\frac{d\mathfrak{A}}{dx} + \frac{1}{2}\frac{d\varpi}{dy}\right)\\
&\quad + \tfrac{2}{3}nh^3E^2(1+E)\frac{d\nabla^2K}{dx} - \tfrac{1}{3}\rho h^3E(1-2E)\frac{d\Omega}{dx}\\
\rho(\ddot{v}-Y) &= 2n\left(\frac{d\mathfrak{B}}{dy} + \frac{1}{2}\frac{d\varpi}{dx}\right)\\
&\quad + \tfrac{2}{3}nh^3E^2(1+E)\frac{d\nabla^2K}{dy} - \tfrac{1}{3}\rho h^3E(1-2E)\frac{d\Omega}{dy}
\end{aligned}\right\}\ \ldots.(53);$$

and the values of T_1, T_2, M, viz.:

$$\left.\begin{aligned}
T_1 &= 4nh\mathfrak{A} + \tfrac{2}{3}nh^3E\left\{\frac{d^2K}{dx^2} + E(3+2E)\nabla^2K\right\} + \tfrac{2}{3}\rho h^3E^2\Omega\\
T_2 &= 4nh\mathfrak{B} + \tfrac{2}{3}nh^3E\left\{\frac{d^2K}{dy^2} + E(3+2E)\nabla^2K\right\} + \tfrac{2}{3}\rho h^3E^2\Omega\\
M &= 2nh\varpi + \tfrac{2}{3}nh^3E\frac{d^2K}{dx\,dy}
\end{aligned}\right\}\ \ldots..(54).$$

The accuracy of these results can be tested by means of (12) and (13); for, if in these equations we substitute the values T_1, T_2, M from (54), and the value of $\ddot{K}$ from (49), we shall reproduce (53).

The boundary conditions at a free edge require that T_1, T_2, M should vanish there; and therefore the conditions to be satisfied at a free edge are, that the right-hand sides of equation (54) should vanish at every point of the free boundary.

If we suppose that the plate is infinite, and that $v=0$, and u is a function of x alone, and that there are no bodily forces, the first of (53) becomes

$$\rho\ddot{u} = 2n\,(1+E)\left(\frac{d^2u}{dx^2} + \tfrac{1}{3}E^2h^2\frac{d^4u}{dx^4}\right);$$

whence, putting $$u \propto e^{pt+\sqrt{x}},$$

we obtain $$\rho p^2 = 2f^2n\,(1+E)(1-\tfrac{1}{3}f^2E^2h^2),$$

which is Lord Rayleigh's result.*

The method employed has the advantage of furnishing a test of the accuracy of our work, and also of the fundamental hypothesis upon which it is based, viz., *that if the surfaces of the plate are not subjected to any pressures or tangential stresses, the three stresses R, S, T are of the order of the square of the thickness.*† We have, first of all, obtained the equations of motion in terms of the resultant edge stresses; we have, in the next place, obtained an expression for the potential energy due to strain, and by means of it have deduced the equations of motion, and the values of the edge stresses in terms of the displacements; we have, finally, substituted the values of the edge stresses in terms of the displacements in our original equations (12), &c., and we find that the equations of motion which we have obtained in terms of the displacements by the variational method are thereby reproduced, as ought to be the case.

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* *Proc. Lond. Math. Soc.*, Vol. xx., equation (45), p. 231.

† [The meaning of this hypothesis may perhaps be made clearer by stating that it supposes that the values of these stresses at any point of the substance of the plate, so far as they depend upon h, h', are of the form

$$u_2 + u_3 + \ldots + u_n + \ldots$$

where u_n is a homogeneous n-tic function of h and h'. *April 17th*, 1890.]

Thursday, December 12th, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Messrs. Bhuban Mohan Bannerji, B.A., F.R. Hist. Soc., Professor of Mathematics, Christian College, Lucknow, India; Ralph Allen Sampson, B.A., St. John's College, Cambridge, Senior Mathematical Lecturer, King's College, London; James MacMahon, B.A., T.C.D., Professor of Mathematics, Cornell University, Ithaca, New York; and A. L. Selby, M.A., Fellow of Merton College, Oxford, were elected members. Professor W. Burnside was admitted into the Society.

The Auditor (Mr. Heppel) made his report; on the motion of Sir James Cockle, F.R.S., V.-P., seconded by Mr. W. J. C. Sharp, he was thanked for the trouble he had taken. On the motion of Mr. Basset, F.R.S., seconded by Dr. Morrice, the Treasurer's Report was then adopted.

The following communications were made:—

On the Radial Vibrations of a Cylindrical Elastic Shell: Mr. Basset.

Note on the 51840 Group: Dr. Morrice.

Mr. Elliott, V.-P., having taken the Chair, the following further papers were read:—

Complex Multiplication Moduli of Elliptic Functions for the Determinants -53 and -61: Prof. G. B. Mathews;

On the Flexure of an Elastic Plate: Prof. H. Lamb; and

Notes on a Plane Cubic and a Conic: Mr. R. A. Roberts.

Dr. Larmor and Mr. Sharp made short communications.

The following presents were received:—

"Educational Times," for December.

"Transactions of the Cambridge Philosophical Society," Vol. XIV., Part IV.

"Mathematics from the 'Educational Times,'" Vol. LI.

"Jahrbuch über die Fortschritte der Mathematik," Band XIX., Heft 1; Jahrgang 1887, Heft 1.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 93 and 94.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 3.

"Journal für die reine und angewandte Mathematik," Band 105, Heft IV.

"Memorie della Regia Accademia di Scienze, Lettere, ed Arti in Modena," *Serie II.*—Vol. VI.; 1888.

"Extrait d'une Lettre adressée à M. Guccia," par M. H.-G. Zeuthen. ("Rendiconti del Circolo Matematico di Palermo.")

"Note sur les Huits Points d'Intersection de Trois Surfaces du Second Ordre," par H.-G. Zeuthen. ("Acta Mathematica.")

"American Journal of Mathematics," Vol. XII., No. 2; Baltimore.

On the Radial Vibrations of a Cylindrical Elastic Shell.

By A. B. Basset, M.A., F.R.S.

[*Read Dec. 12th*, 1889.]

The usual theory of thin plates and shells assumes that the three stresses R, S, T may be treated as zero; and in a previous paper* I pointed out, that this supposition is legitimate, only in the case in which these quantities are at least of the order of the square of the thickness of the plate or shell. I also showed that, by means of certain results given in a paper by Lord Rayleigh, it could be easily proved that this supposition is true in the case of a plane plate, which is not under the influence of forces applied to its surface.

The question whether R, S, T *may be treated as zero* (for they cannot actually be zero except in certain very special cases) is of the utmost importance in the theory of the free vibrations of thin curved shells. If these stresses may be treated as zero, it is possible to obtain the correct expression for the potential energy, and the correct equations of motion, as far as the terms involving the cube of the thickness; but, if they may not be treated as such, a satisfactory theory of thin shells seems almost hopeless. For, although the results obtained by considering (as Mr. Love has done) that part of the potential energy which is *solely* due to the extension of the middle surface, are correct as far as the terms involving the thickness are concerned; yet, inasmuch as there are reasons for thinking that the graver tones of a bell depend principally upon flexure rather than extension, it is essential to take the term in h^3 into consideration.

A perfectly rigorous solution of any question relating to the vibrations of cylindrical or spherical shells, might be obtained by means of the general equations of motion of an elastic solid, since these

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* Ante, p. 33.

equations contain within themselves the complete solution of every conceivable problem; but the mathematical difficulties of integrating these equations, except in special cases, are so great, that an investigation conducted by means of them would be very laborious and complicated. The radial vibrations of an infinitely long cylindrical shell can, however, be investigated by means of these equations without much difficulty as far as the term in the period which involves h^2, where $2h$ is the thickness of the shell; and, by comparing the result obtained by this method with the one furnished by the theory of thin plates, we shall obtain evidence respecting the legitimacy of the assumption on which the latter theory is based.

The vibrations which we are about to consider are exclusively normal, and the displacement w is a function of r alone, and satisfies the differential equation

$$(m+n)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{w}{r^2}\right)=\rho\ddot{w} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

as can easily be seen by putting $u=v=0$ in the general equations of motion of an elastic solid.

Putting $$w=We^{ipt}, \quad p^2\rho/(m+n)=\alpha^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

(1) becomes $$\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}+\left(\alpha^2-\frac{1}{r^2}\right)W=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the solution of which is

$$W=AJ_1(\alpha r)+BY_1(\alpha r) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where J and Y are the two kinds of Bessel's functions. Since the tangential stress parallel to the middle surface $r=a$ is obviously zero in this species of motion, the only surface condition is $E=0$, or

$$\frac{dW}{dr}+\frac{EW}{r}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $$E=\frac{m-n}{m+n};$$

whence, putting $x=\alpha r$, (5) becomes

$$A\{xJ_1'(x)+EJ_1(x)\}+B\{xY_1'(x)+EY_1(x)\}=0 \ldots\ldots(6),$$

when $$r=a\pm h.$$

This equation enables us to determine the frequency, and we shall proceed to evaluate it as far as the term involving h^2.

Denoting the coefficients of A and B in (6) by $\phi(x)$ and $\chi(x)$,

where x is now written for aa, the equation for the frequency is

$$\frac{\varphi(x+ah)}{\varphi(x-ah)} = \frac{\chi(x+ah)}{\chi(x-ah)},$$

whence, neglecting powers of h higher than h^2, we have

$$\phi\chi' - \chi\phi' + \tfrac{1}{2}a^2h^2\{\varphi''\chi' - \chi''\varphi' + \tfrac{1}{3}(\phi\chi''' - \chi\phi''')\} = 0 \quad \ldots\ldots(7).$$

In evaluating this equation the following properties of Bessel's functions will be useful, which can easily be proved by means of the fundamental equation which these quantities satisfy.

Let $$H_s = J_s' Y_s - Y_s' J_s,$$

then it can be proved that

$$\left.\begin{aligned}
J_s'' Y_s - Y_s'' J_s &= -H_s/x, \\
J_s'' Y_s' - Y_s'' J_s' &= H_s(1 - s^2/x^2), \\
J_s''' Y_s - Y_s''' J_s &= H_s\left(\frac{s^2+2}{x^2} - 1\right), \\
J_s''' Y_s' - Y_s''' J_s' &= \frac{H_s}{x}\left(\frac{3s^2}{x^2} - 1\right), \\
J_s^{\text{iv}} Y_s - Y_s^{\text{iv}} J_s &= -\frac{2H_s}{x}\left(\frac{3s^2+3}{x^2} - 1\right), \\
J_s^{\text{iv}} Y_s' - Y_s^{\text{iv}} J_s' &= -H_s\left(\frac{s^4+11s^2}{x^4} - \frac{2s^2+3}{x^2} + 1\right),
\end{aligned}\right\} \quad \ldots\ldots(8),$$

to which may be joined

$$J_s''' Y_s'' - Y_s''' J_s'' = H_s\left(1 - \frac{2s^2+1}{x^2} + \frac{s^4-s^2}{x^4}\right) \quad \ldots\ldots(9).$$

It is also known that xH_s is independent of x.

Dropping the suffix 1, which is no longer necessary, we have

$$\begin{aligned}
\phi &= xJ' + EJ, \\
\phi' &= xJ'' + (1+E)J', \\
\phi'' &= xJ''' + (2+E)J'', \\
\phi''' &= xJ^{\text{iv}} + (3+E)J''';
\end{aligned}$$

whence $$\phi\chi' - \chi\phi' = H(1 - E^2 - x^2) \quad \ldots\ldots(10).$$

From this equation we see that, to a first approximation,

$$x^2 = 1-E^2 \quad \ldots\ldots(11);$$

whence, restoring the values of x and E, we obtain

$$p^2 = \frac{4mn}{\rho a^2(m+n)} \quad \ldots\ldots(12),$$

which is a known result.

Again,

$$\phi''\chi' - \chi''\phi' = x^3(J'''Y'' - Y'''J'') + x(1+E)(J'''Y' - Y'''J') + (1+E)(2+E)(J''Y' - Y''J')$$

$$= -H(3-x^2) + H(1+E)\left(\frac{3}{x^2}-1\right) + H(1+E)(2+E)\left(1-\frac{1}{x^2}\right)$$

$$= \frac{H}{x^2}\left[x^4 - 3x^2 + 1 - E^2 + (1+E)^2x^2\right] \quad \ldots\ldots(13).$$

Also,

$$\phi\chi''' - \chi\phi''' = x^4(Y^{iv}J' - J^{iv}Y') + xE(Y^{iv}J - J^{iv}Y) + x(3+E)(Y'''J' - J'''Y') + E(3+E)(Y'''J - J'''Y)$$

$$= \frac{H}{x^2}\left\{x^4 + x^2(E^2+2E-2) + 3(1-E^2)\right\} \quad \ldots\ldots(14).$$

Substituting from (10), (13), (14), in (7), we obtain

$$x^2(1-E^2) - x^4 + a^2h^2\left\{\tfrac{2}{3}x^4 - 2x^2 + \tfrac{2}{3}(1+E)^2x^2 + 1 - E^2\right\} = 0.$$

Now $x = a\alpha$; we may also, in the term involving h^2, substitute the approximate value of x from (11); we thus obtain

$$a^2(1-E^2) - a^4\alpha^2 + \tfrac{1}{3}h^2(1-E^2)(1+4E) = 0;$$

whence

$$p^2 = \frac{4mn}{\rho a^2(m+n)}\left\{1 + \frac{h^2}{3a^2}(1+4E)\right\} \quad \ldots\ldots(15),$$

from which it appears that the pitch rises as the thickness increases.

We shall now prove that the same result is given by the theory of thin plates, in which R is treated as zero.

Let

$$d\Omega = dx\,d\phi;$$

then, if T and W be the kinetic and potential energies,

$$T = \tfrac{1}{2}\rho \iiint_{-h}^{h} \dot{w}'^2 (a+h')\, dh' d\Omega,$$

where $a+h'$, ϕ are the coordinates of any point of the shell.

Now $$w' = w + h'\left(\frac{dw}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2w}{dr^2}\right),$$

where the brackets denote the values of the differential coefficients when $r=a$. Also, if we suppose that R is at most of the order of the square of the thickness, and neglect higher powers than h^3, in the expressions for T and W, we may write

$$\frac{dw}{dr} = \sigma_2 = -Ew/r,$$

$$\frac{d^2w}{dr^2} = E(1+E)\, w/r^2;$$

accordingly $$w' = \left\{1 - \frac{Eh'}{a} + \tfrac{1}{2}E(1+E)\frac{h'^2}{a^2}\right\} w;$$

whence $$T = \rho h \iint \left\{1 + \frac{h^2 E}{3a^2}(2E-1)\right\} \dot{w}^2 a\, d\Omega \quad \ldots\ldots\ldots\ldots (16).$$

Again, $$W = \iiint_{-h}^{h} \left\{\tfrac{1}{2}(m+n)\Delta'^2 - 2n\sigma_1\sigma_2\right\} r\, dh' d\Omega$$

$$= n(1+E) \iiint_{-h}^{h} \frac{w'^2}{r}\, dh' d\Omega$$

$$= 2nh(1+E) \iint \left\{1 + \frac{h^2}{3a^2}(1+E)(1+2E)\right\} \frac{w^2}{a^2}\, a\, d\Omega \ldots (17);$$

whence

$$\rho a^2 p^2 \left\{1 + \frac{h^2 E}{3a^2}(2E-1)\right\} = 2n(1+E)\left\{1 + \frac{h^2}{3a^2}(1+E)(1+2E)\right\},$$

or $$p^2 = \frac{4mn}{\rho a^2 (m+n)}\left\{1 + \frac{h^2}{3a^2}(1+4E)\right\} \quad \ldots\ldots\ldots\ldots (18),$$

which agrees with our former result.

Although the agreement of the results obtained by the two theories in this particular case does not, of course, prove that R may be treated as zero in every conceivable case that may arise, yet the inference is that it may be treated as such, provided the surfaces of the shell are free from external pressures and tangential stresses, and provided also that the solution is not carried to a higher degree of approxima-

tion than terms in h^3 in the expressions for the kinetic and potential energies, and terms in h^2 in the period.

A similar method might be employed to investigate the radial vibrations of a complete spherical shell, for the equation of motion is

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+a^2-\frac{9}{4r^2}\right)(wr^{\frac{1}{2}})=0,$$

the solution of which is

$$w=r^{-\frac{1}{2}}\left\{AJ_{\frac{1}{2}}(ar)+BY_{\frac{1}{2}}(ar)\right\},$$

and evaluating by means of (8).

Note on a Quaternary Group of 51840 *Linear Substitutions.*

By Dr. Morrice.

[*Read December 12th*, 1889.]

Part I.

I give in Part I. a few fundamental data concerning the transformation of the double functions which will be useful to those who have not followed the development of the subject by Witting* and Maschke† in the *Mathematische Annalen*.

We have the periods $\omega_{11}, \omega_{12}, \omega_{13}, \omega_{14},$
$\omega_{21}, \omega_{22}, \omega_{23}, \omega_{24}$
of the hyper-elliptic integrals

$$u_1=\int\frac{\alpha+\beta z}{\sqrt{f(z)}}dz, \quad u_2=\int\frac{\gamma+\delta z}{\sqrt{f(z)}}dz,$$

and we transform by means of the relations

$$v_1=\frac{u_1\omega_{22}-u_2\omega_{12}}{p_{12}}, \quad v_2=\frac{-u_1\omega_{21}+u_2\omega_{11}}{p_{12}},$$

where p_{ik} stands for $\omega_{1i}\omega_{2k}-\omega_{1k}\omega_{2i}$.

The new periods are

$$1, \quad 0, \quad \tau_{11}=\frac{p_{23}}{p_{12}}, \quad \tau_{12}=\frac{p_{42}}{p_{12}},$$

$$0, \quad 1, \quad \tau_{21}=\frac{p_{13}}{p_{12}}, \quad \tau_{22}=\frac{p_{14}}{p_{12}},$$

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* Witting, Inaugural Dissertation, Dresden, 1887, and *Math. Ann.*, Band 29.

† Maschke, *Math. Ann.*, Band 33, Heft 3.

and the relation, $p_{13} = p_{42}$,

which exists between the old periods, becomes

$$\tau_{12} = \tau_{21}.$$

The double ϑ-function of the first order is

$$\vartheta_{\left|\begin{smallmatrix} g_1, g_2 \\ h_1, h_2 \end{smallmatrix}\right|}(v_1, v_2) = e^{2i\pi(g_1h_1+g_2h_2)} \sum_{-\infty}^{+\infty} n_1 n_2 (-1)^{n_1h_1+n_2h_2} e^{i\pi[(2n_1+g_1)v_1+(2n_2+g_2)v_2} \times e^{2i\pi\phi(2n_1+g_1,\, 2n_2+g_2)},$$

where $\phi(n_1, n_2) = r_{11}n_1^2 + 2r_{12}n_1n_2 + r_{22}n_2^2$.

We have now to consider the group of linear transformations of the periods which leave the characteristic

$$\begin{vmatrix} 0, & 0 \\ 0, & 0 \end{vmatrix}$$

unaltered. Witting gives five from which any others may be generated by iteration and combination:

	M	N	P	Q	R
$\omega'_{i1} =$	ω_{i3}	ω_{i1}	$\omega_{i2}+\omega_{i3}$	ω_{i1}	$-\omega_{i1}+\omega_{i3}$
$\omega'_{i2} =$	ω_{i3}	$-\omega_{i1}-\omega_{i4}$	ω_{i3}	ω_{i4}	ω_{i}
$\omega'_{i3} =$	$-\omega_{i1}$	$\omega_{i1}+\omega_{i2}+\omega_{i3}+\omega_{i4}$	$-\omega_{i1}-\omega_{i3}$	ω_{i3}	$-\omega_{i3}$
$\omega'_{i4} =$	ω_{i4}	$-\omega_{i1}+\omega_{i3}$	$\omega_{i1}-\omega_{i2}-\omega_{i3}-\omega_{i4}$	$-\omega_{i2}$	$\omega_{i3}+\omega_{i4}$

$$i = 1, 2.$$

In connexion with the theory of transformation of order k, there have recently been established functions analogous to the sigma-functions for elliptic integrals; *i.e.*, we have a set of functions

$$X_{\alpha\beta}(v_1, v_2;\ \tau_{11}, \tau_{12}, \tau_{22})$$

$$= p_{12}^{\frac{1}{2}(k-1)} \frac{e^{2i\pi\Phi(v_1,v_2)}}{\vartheta(\tau_{11}, \tau_{12}, \tau_{22})^k} \sum_{-\infty}^{+\infty} e^{(i\pi/k)\phi(kn_1+\alpha,\, kn+\beta)+2i\pi[(kn_1+\alpha)v_1+(kn_2+\beta)v_2]},$$

$$(\alpha, \beta = 0, 1, \ldots k-1),$$

where $\Phi(v_1, v_2) = A_{11}v_1^2 + 2A_{12}v_1v_2 + A_{22}v_2^2$,

$$A_{11} = -\frac{1}{5}\sum_{1}^{10}\frac{\partial \log\vartheta}{\partial\tau_{11}},$$

$$A_{12} = -\frac{1}{10}\sum_{1}^{10}\frac{\partial \log\vartheta}{\partial\tau_{12}},$$

$$A_{22} = -\frac{1}{5}\sum_{1}^{10}\frac{\partial \log\vartheta}{\partial\tau_{22}}.$$

For a transformation of the periods which leaves the characteristic

$$\begin{vmatrix} 0, & 0 \\ 0, & 0 \end{vmatrix}$$

unaltered, these $X_{\alpha\beta}$ functions are transformed into linear functions of themselves, with merely numerical coefficients. The particular case under consideration is that where $k = 3$.

We choose the four functions

$$\begin{aligned} z_1 &= X_{01} - X_{02}, \\ z_2 &= X_{10} - X_{20}, \\ z_3 &= X_{11} - X_{22}, \\ z_4 &= X_{12} - X_{21}, \end{aligned}$$

and establish the transformations which they undergo when the ω-periods undergo the transformations M, N, P, Q, R,

$$M\begin{cases} z_1' = \frac{i}{\sqrt{3}}(z_1 + z_3 - z_4), \\ z_2' = \frac{i}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_2, \\ z_3' = \frac{i}{\sqrt{3}}(z_1 + \epsilon^2 z_3 - \epsilon z_4), \\ z_4' = \frac{i}{\sqrt{3}}(-z_1 - \epsilon z_3 + \epsilon^2 z_4), \end{cases} \qquad N\begin{cases} z_1' = \frac{-i}{\sqrt{3}}(\epsilon - \epsilon^2)\, z_1, \\ z_2' = \frac{-i}{\sqrt{3}}(\epsilon z_2 + \epsilon^2 z_3 + z_4), \\ z_3' = \frac{-i}{\sqrt{3}}(\epsilon^2 z_2 + \epsilon z_3 + z_4), \\ z_4' = \frac{-i}{\sqrt{3}}(z_2 + z_3 + z_4), \end{cases}$$

$$P\begin{cases} z_1' = \frac{-1}{\sqrt{3}}(z_1 + \epsilon z_3 - \epsilon^2 z_4), \\ z_2' = \frac{-1}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_2, \\ z_3' = \frac{-1}{\sqrt{3}}(\epsilon^2 z_1 + \epsilon^2 z_3 - \epsilon^2 z_4), \\ z_4' = \frac{-1}{\sqrt{3}}(-\epsilon z_1 - z_3 + \epsilon^2 z_4), \end{cases} \qquad Q\begin{cases} z_1' = \frac{-1}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_1, \\ z_2' = \frac{-1}{\sqrt{3}}(z_2 + z_3 + z_4), \\ z_3' = \frac{-1}{\sqrt{3}}(z_2 + \epsilon^2 z_3 + \epsilon z_4), \\ z_4' = \frac{-1}{\sqrt{3}}(z_2 + \epsilon z_3 + \epsilon^2 z_4), \end{cases}$$

$$R\begin{cases} z_1' = -z_3, \\ z_2' = z_2, \\ z_3' = -z_1, \\ z_4' = -z_4, \end{cases} \qquad (\epsilon = e^{\frac{2i\pi}{3}}).$$

Part II.

It appears natural to inquire what transformations are common to the periods, on the one hand, and the four Jacobian functions z_1, z_2, z_3, z_4, on the other. Is it possible to find a sub-group of transformations common to the two groups, such that to the transformations of the ω's correspond the same transformations taken in a different order?

I know of no general method for answering these questions, and after searching through some hundreds of the matrices, I only found three common to the two groups; but these three have a simple property which appears worthy of observation.

To the matrix (which occurs in both groups)

$$S(\omega) \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right|$$

in the ω-group, corresponds the matrix

$$S(z) \left| \begin{array}{cccc} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{array} \right|$$

in the z-group.

To the matrix (which occurs in both groups)

$$T(\omega) \left| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right|$$

in the ω-group, corresponds the matrix

$$T(z) \left| \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array} \right|$$

in the z-group; for the sake of simplicity certain numerical factors of the matrices are disregarded.

Now when the ω's are subjected to the matrix

$$S.T(\omega) \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

the z's are transformed by the same matrix, *i.e.*, are covariant for this matrix.

Moreover, we find by actual composition

$$S(\omega)\ T(z) = T(\omega)\ S(z) = \{T.S(\omega)\}^{\frac{1}{4}} = \{T.S(z)\}^{\frac{1}{4}},$$
$$S(z)\ T(\omega) = T(z)\ S(\omega) = \{S.T(\omega)\}^{\frac{1}{4}} = \{S.T(\omega)\}^{\frac{1}{4}},$$
$$S(\omega)\ S(z)\ S^{-1}(\omega) = T(z).$$

In terms of Witting's matrices,

$$S = Q^2 M(\omega),$$
$$T = M^2 (QMRM^2)^2 (\omega).$$

Herr Burkhardt's *résumé* of Klein's lectures in *Mathematische Annalen*, Band xxv., Heft 2, suggests that there may be some geometrical explanation of the curious simplicity of the sub-group in question.

Notes on the Plane Cubic and a Conic. *By* R. A. Roberts.

[*Read Dec. 12th*, 1889.]

I commence by showing that a plane cubic and a conic can be reduced to the forms

$$U = ax^3 + \beta y^3 + \gamma z^3 + \delta u^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$V = ax^3 + by^3 + cz^3 + du^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

in a single way, where x, y, z, u are four lines in the plane, and α, β,

γ, δ, a, b, c, d are constants. These may thus be considered as canonical forms for a cubic and a conic.

To prove this, I introduce the consideration of another conic Σ touching the lines x, y, z, u. This conic possesses the property that if we substitute differential symbols in its tangential equation for λ, μ, ν, respectively, and operate on both the equations of the cubic and conic, the results will vanish. For, if we operate in this way on the power of any line, L^n say, the result will be proportional to L^{n-2} multiplied by the condition that L and Σ should touch. Consequently, since the cubic and conic only involve powers of x, y, z, u, and the conic Σ touches these lines, the result just stated must have place. Now, if a conic be such that the result of substituting differential symbols in its tangential equation and operating on a cubic vanishes, it must satisfy three invariant relations with the cubic; for such a result must be linear in x, y, z, $Ax+By+Cz$, say; and, in order that this should vanish identically, we should have

$$A = B = C = 0.$$

Hence the conic Σ satisfies three invariant relations with the cubic U, and one with the conic V. But these four conditions are geometrically expressed in the fact that it touches the four lines x, y, z, u, when U and V are written in the forms (1), (2). Thus, if U and V are written in any other forms, and we express that Σ satisfies certain three conditions with U, and one condition with V, and find then that these four conditions, taken together, are equivalent to expressing that Σ touches four fixed lines, we infer that these four lines are the lines x, y, z, u, when U and V are written in the forms (1), (2); and that this is so I proceed to prove.

If the cubic be written in one of its own canonical forms, namely,

$$U \equiv x^3+y^3+z^3+6mxyz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and the conic V is then at the same time

$$V = (a, b, c, f, g, h)(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the results of substituting differential symbols in the tangential equation of a conic Σ $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$ and operating on the cubic U and the conic V are, respectively,

$$(A+2mF)x+(B+2mG)y+(C+2mH)z,$$

$$Aa+Bb+Cc+2Ff+2Gg+2Hh;$$

and, in order that these should identically vanish, we must have

$$A+2mF=0, \quad B+2mG=0, \quad C+2mH=0,$$

$$Aa+Bb+Cc+2Ff+2Gg+2Hh=0,$$

showing that Σ must be of the form

$$A(m\lambda^2-\mu\nu)+B(m\mu^2-\nu\lambda)+C(m\nu^2-\lambda\mu)=0 \quad \ldots\ldots\ldots(5),$$

where $$A(ma-f)+B(mb-g)+C(mc-h)=0 \ldots\ldots\ldots\ldots\ldots(6).$$

Now, subject to these conditions, Σ evidently touches the four lines determined by the equations

$$\frac{m\lambda-\mu\nu}{ma-f}=\frac{m\mu^2-\nu\lambda}{mb-g}=\frac{m\nu^2-\lambda\mu}{mc-h} \quad \ldots\ldots\ldots\ldots\ldots(7),$$

and these must therefore be the four lines involved in the canonical forms (1), (2). Their equation may be obtained by taking the equation of Σ in x, y, z coordinates, which may be written

$$A^2(4myz-x^2)+B^2(4mzx-y^2)+C^2(4mxy-z^2),$$

and $$2BC(yz+4m^2x^2)+2CA(zx+4m^2y^2)+2AB(xy+4m^2z^2)=0 \ldots(8),$$

and forming its envelope subject to the condition

$$A(ma-f)+B(mb-g)+C(mc-h)=0.$$

This gives

$$\begin{aligned}
&(ma-f)^2 mx(3mxyz-y^3-z^3-m^3x^3)\\
&+(mb-g)^2 my(3mxyz-z^3-x^3-m^3y^3)\\
&+(mc-h)^2 mz(3mxyz-x^3-y^3-m^3z^3)\\
&+(mb-g)(mc-h)\{2(m^4-m)y^2z^2+x^2yz(1-4m^3)+m^3x(x^3+y^3+z^3)\}\\
&+(mc-h)(ma-f)\{2(m^4-m)z^2x^2+y^2zx(1-4m^3)+m^3y(x^3+y^3+z^3)\}\\
&+(ma-f)(mb-g)\{2(m^4-m)x^2y^2+z^2xy(1-4m^3)+m^3z(x^3+y^3+z^3)\}\\
&\qquad\qquad =0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).
\end{aligned}$$

This equation, it may be observed, is of the second degree in the coefficients of the conic, and of the sixth in those of the cubic.

We may notice that, if polar conics of U be described to have double contact with V, the chords of contact corresponding to the four solutions are the lines x, y, z, u in the forms (1), (2). For a polar conic of U is

$$ax_1x^2+\beta y_1y^2+\gamma z_1z^2+\delta u_1u^2=0,$$

and this will have double contact with V at points lying on $u=0$,

if we take

$$\alpha x_1 = a, \quad \beta y_1 = b, \quad \gamma z_1 = c,$$

and similarly in the case of the other lines x, y, z.

The coefficients α, β, γ, δ can be found as the roots of a biquadratic equation as follows. The diagonals of the quadrilateral x, y, z, u can, it is easy to see, be written in the form

$$J \equiv x^3+y^3+z^3+u^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

on the supposition $\qquad x+y+z+u = 0,$

so that $U-\lambda J$ is

$$(\alpha-\lambda)x^3+(\beta-\lambda)y^3+(\gamma-\lambda)z^3+(\delta-\lambda)u^3 = 0,$$

and the invariant S of this cubic is proportional to

$$(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda)$$

(see Salmon's *Higher Plane Curves*, Art. 239, 3rd ed.), which thus being equated to zero will give the required biquadratic. In order to find the equation of J when the cubic and conic are written in the forms (3), (4), we observe that the diagonals are the common self-conjugate triangle of all the conics inscribed in the quadrilateral. Hence, according to the theory of conics, their equation will be the Jacobian of three conics of the system; for instance, if we take the three conics to be those whose tangential equations are

$$\left.\begin{aligned}(ma-f)(m\mu^2-\nu\lambda)-(mb-g)(m\lambda^2-\mu\nu) &= 0,\\ (mb-g)(m\nu^2-\lambda\mu)-(mc-h)(m\mu^2-\nu\lambda) &= 0,\\ (mc-h)(m\lambda^2-\mu\nu)-(ma-f)(m\nu^2-\lambda\mu) &= 0,\end{aligned}\right\} \quad \ldots\ldots\ldots(11),$$

J will be the Jacobian of

$$\left.\begin{aligned}&\{(mb-g)x+(ma-f)y\}^2+4m^2(ma-f)(mb-g)z^2\\ &\qquad -4mz\{(ma-f)^2x+(mb-g)^2y\} = 0,\\ &\{(mc-h)y+(mb-g)z\}^2+4m^2(mb-g)(mc-h)x^2\\ &\qquad -4mx\{(mb-g)^2y+(mc-h)^2z\} = 0,\\ &\{(ma-f)z+(mc-h)x\}^2+4m^2(ma-f)(mc-h)y^2\\ &\qquad -4my\{(ma-f)^2x+(mc-h)^2z\} = 0,\end{aligned}\right\} \quad \ldots\ldots(12).$$

When a conic is written in the form (2) it is easily seen that the extremities of each of the three diagonals of the quadrilateral x, y, z, u are conjugate with respect to the curve, so that, if it breaks up into

right lines, one line can be assumed arbitrarily, and then the other will be completely determined, namely, will pass through the three fourth harmonics on each of the diagonals. Hence, if one line is the line at infinity, the other will pass through the middle points of the diagonals. Now, when a cubic is written in the form (1), its Hessian is

$$\beta\gamma\delta yzu+\gamma\delta\alpha zux+\delta\alpha\beta uxy+\alpha\beta\gamma xyz=0,$$

that is, passes through the six points of intersection of the lines x, y, z, u. Hence we see that the problem, to describe a quadrilateral to have its six intersections of sides on a cubic so that the middle points of the diagonals should lie on a given line, admits of three solutions; for there are three cubics which have a given cubic for a Hessian.

When a cubic and a conic are written in the forms (1), (2), a certain covariant conic is also expressible linearly in terms of the squares of x, y, z, u. This covariant may be found as follows. Writing a line in the form

$$\lambda x+\mu y+\nu z+\rho u=0,$$

its polar conic with regard to the cubic, namely, the envelope of the polars with regard to the cubic of the points on the line, is found to be

$$\Sigma\gamma\delta zu(\lambda-\mu)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Now the tangential equation of V is

$$\Sigma cd(\lambda-\mu)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Hence, if we substitute differential symbols $\frac{d}{d\lambda}$, $\frac{d}{d\mu}$, $\frac{d}{d\nu}$, $\frac{d}{d\rho}$ for x, y, z, u in the former equation, and operate on the latter, we get

$$\Sigma ab\gamma\delta(\lambda-\mu)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

which is therefore a contravariant conic.

But this latter is the tangential equation of

$$abcdx^2+\beta cday^2+\gamma dabz^2+\delta abcu^2=0 \quad \ldots\ldots\ldots\ldots(16).$$

If this conic be written

$$a'x^2+b'y^2+c'z^2+d'u^2=0,$$

we have $$aa' : bb' : cc' : dd' = \alpha : \beta : \gamma : \delta,$$

which shows that V and the covariant (16) are reciprocally related with regard to the cubic. This will be readily seen when the cubic

and conic are written in the forms (3), (4). If the tangential equation of V is then

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0,$$

the tangential equation of the contravariant is

$$(m^2A-F, m^2B-G, m^2C-H, mA-m^2F, mB-m^2G, mC-m^2H)(\lambda, \mu, \nu)^2 = 0 \quad \ldots\ldots(17).$$

From these equations it is readily seen that the conic V and the contravariant will coincide if either is of the form

$$A(\lambda^2+2\theta\mu\nu)+B(\mu^2+2\theta\nu\lambda)+C(\nu^2+2\theta\lambda\mu) = 0,$$

where $$\theta^2-2m^2\theta+m = 0,$$

that is, is a derived conic of either of the curves of the third class

$$\lambda^3+\mu^3+\nu^3+6\theta\lambda\mu\nu = 0 \quad \ldots\ldots(18).$$

If one of the coefficients of V in (2) vanish, d say, the contravariant becomes, from (15), after dividing by δ,

$$bc\alpha(\mu-\nu)^2+ca\beta(\nu-\lambda)^2+ab\gamma(\lambda-\mu)^2 \equiv 0,$$

that is, it breaks up into factors, the two points corresponding to which lie on the line $u = 0$. Now x, y, z form a triangle inscribed in the Hessian, so that the points where the sides meet the curve again lie on a line; hence, if it is possible that such a triangle should be self-conjugate with regard to a conic, the invariant relation connecting the cubic and the conic is found by taking the discriminant of (17). This relation is therefore of the sixth degree in the coefficients of the cubic, and the third degree in the tangential coefficients of the conic.

We now proceed to show how to obtain the conditions that a conic should be circumscribed about or inscribed in a triangle which is inscribed in the Hessian, so that the points where the sides meet the curve again are collinear. Suppose that the cubic is

$$ax^3+\beta y^3+\gamma z^3+\delta u^3 = 0,$$

and if a conic circumscribes the triangle xyz, we may write it

$$fyz+gzx+hxy = 0.$$

Substituting the differential symbols $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{d\nu}$ for x, y, z, respec-

tively, in the latter equation, and operating on the tangential equation of the polar conic of a point, namely,

$$\Sigma\gamma\delta zu\,(\lambda-\mu)^2 = 0,$$

we get a covariant conic which is found to be

$$\delta u\,(fax+g\beta y+h\gamma z) = 0,$$

that is, it breaks up into factors in the case under consideration. When the cubic and conic are written in the forms (3), (4), the covariant is

$$a\,(yz-m^3x^3)+b\,(zx-m^3y^3)+c\,(xy-m^3z^3)+2f\,(m^3yz-mx^3)$$
$$+2g\,(m^3zx-my^3)+2h\,(m^3xy-mz^3) = 0 \ldots\ldots\ldots\ldots (19).$$

It is thus linear in the coefficients of the conic, and of the second degree in those of the cubic, so that its discriminant, or the condition required, is of the third degree in the coefficients of the conic and of the sixth degree in those of the cubic.

For example, if the circumscribing conic is a circle which cuts a given circle orthogonally, the foregoing relation shows that its centre lies on a given curve of the third degree. The conic and the covariant (19), it is easy to see, are reciprocally related with regard to the cubic; in fact, one passes through the points on the Hessian corresponding to those in which the other intersects it. This follows from the fact that, for points on the Hessian, the polar conics break up into pairs of lines intersecting at the corresponding points. Hence we see that the result given above follows at once; for, when three points on the Hessian are collinear, the corresponding points form a triangle whose sides meet the curve again in three points on a line.

It may be noticed from this that, if a conic touch the Hessian in some point, its covariant (19) will touch the Hessian in the corresponding point.

We now proceed to find the conditions that a cubic and a conic can be reduced to the forms

$$U \equiv ax^3+\beta y^3+\gamma z^3+\delta u^3 = 0,$$

$$V \equiv l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy = 0.$$

Now, the tangential equation of V being

$$l\mu\nu+m\nu\lambda+n\lambda\mu = 0,$$

and the polar conic of any point on the line u being

$$\alpha x'x^2+\beta y'y^2+\gamma z'z^2=0,$$

the result of operating with one upon the other vanishes. Again, the polar conic of u with respect to U is

$$\beta\gamma yz+\gamma\alpha zx+\alpha\beta xy=0,$$

which circumscribes a triangle circumscribed about V; therefore the invariant relation $\Theta^2=4\Delta\Theta'$ is satisfied between these two conics (see Salmon's *Conics*, Art. 376).

Now, in the forms (3), (4), let the line u be

$$\lambda x+\mu y+\nu z=0;$$

then, since the result of operating on the polar conic of x, y, z is

$$(A+2mF)\,x+(B+2mG)\,y+(C+2mH)\,z=0,$$

we must have $A+2mF : B+2mG : C+2mH=\lambda : \mu : \nu$.

We then form the invariant relation

$$\Theta^3-4\Delta\Theta'=0$$

between the conic and the polar conic of

$$\lambda x+\mu y+\nu z=0,$$

namely, $\lambda^2(yz-m^2x^2)+\mu^2(zx-m^2y^2)+\nu^2(xy-m^2z^2)$

$$+2\mu\nu(m^2yz-mx^2)+2\nu\lambda(m^2zx-my^2)+2\lambda\mu(m^2xy-mz^2)=0,$$

and find $m^4(\lambda^2+\mu^2+\nu^2)+(1-4m^3)\,\lambda\mu\nu$

$$+2(m-m^4)\left\{F(m\lambda^2-\mu\nu)+G(m\mu^2-\nu\lambda)+H(m\nu^2-\lambda\mu)\right\}=0,$$

or $m(\lambda^2+\mu^2+\nu^2)+(1-4m^3)\,\lambda\mu\nu$

$$-(m-m^4)(A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu)=0 \quad \ldots(20).$$

On the Flexure of an Elastic Plate. *By* HORACE LAMB, M.A., F.R.S.

[*Read Dec. 12th*, 1889.]

The following paper, which was written out in substance more than a year ago, is an attempt to put in as simple and straightforward a form as possible the theory of the flexure of a plane elastic plate of uniform isotropic substance, with comments and illustrations designed to elucidate points of difficulty. I should hardly venture to submit it to the Mathematical Society, except for a certain novelty in the treatment of the boundary conditions, where attention is drawn to a point of some importance not hitherto, I believe, noticed. It is hoped that the method here given may help to remove some of the obscurity which, if we may judge from some recent writings, is still held to attach to this part of the subject.

In a future paper I hope to discuss the case of a curved plate or shell, with special reference to the points in controversy between Mr. A. E. H. Love and Lord Rayleigh.*

1. A few remarks on the nature of the problem may be in place at the outset. It is one of a large class in mathematical physics, where the difficulties of an exact solution are evaded by means of special assumptions based either on observation or on a sort of induction from the known results in such particular cases as admit of rigorous treatment. As instances, we may cite the flexure of a stretched membrane, the theory of "long" waves in canals, the conduction of electricity in a wire of varying section, and so on. In many of these questions, it is true, the auxiliary assumptions required are of so obvious a character that no dispute about them is possible, and it is therefore not always thought necessary to state them expressly. The present problem is, however, by far the most difficult of its class, and, although it has been the subject of many elaborate investigations, there cannot be said to be at present any general agreement as to what is the proper foundation for the theory. Without entering into a detailed criticism of the investigations referred to, we may remark that many of them,† at least as ordinarily presented, appear to set undue

* See *Phil. Trans.*, 1888 (A), pp. 491–546, and *Proc. Roy. Soc.*, Dec. 13, 1888, p. 105.

† More especially those which start from the general equations of elasticity.

limits to the scope of the theory by restrictions as to the continuity of the distribution of impressed force, and the magnitude of its rate of variation from point to point of the substance. For example, it can make no sensible difference to the general form assumed by the plate whether a given force be applied as a pressure on one face or as a tension on the other, or whether it be a bodily force acting on a small volume of the substance—whether, again, it be diffused and continuous, or concentrated and discontinuous, although the distribution of strain in the immediate neighbourhood may be very different in the several cases.

It appears most natural to regard as the object of the theory the determination of the general form of the middle surface, ignoring such minute features as may arise from the particular mode of application of the external forces, in the manner indicated. With this view we shall understand by the "deflection" at any point of the middle surface the mean deflection over an area including this point, whose dimensions are of the same order as the thickness of the plate. By an "element" of the plate, we shall understand the portion cut off by a cylindrical surface drawn through the contour of such an area normal to the middle surface. In like manner, by an element of a normal section we shall understand the portion corresponding to a linear element on the middle surface, of length comparable with the thickness.

This being premised, it will appear that a complete theory can be based on the following principle:

If the external forces on each element of the plate are by themselves in equilibrium, the deflection will be everywhere zero, and the stresses across any element of a normal section will be in equilibrium.

By the method of superposition of stresses and strains, this is equivalent to the following:

The external forces on an element of the plate are sufficiently represented by their force- and couple-resultants; *i.e.*, we need not attend to the particular way in which the forces are distributed within the element. The same applies to the stress across an element of a normal section, and to the applied force on an element of the boundary.

2. It is necessary, as various writers have insisted, to distinguish between the state of things in the main body of the plate and that which obtains near the edges. As regards the former, it is easy to show, on the above principle, that the essential features of the deformation can be represented by the superposition of two particular

states of strain, to be described. The plan of the following investigation is, first, to ascertain the distribution of applied force which would accurately maintain the resulting strains; and then to show that, if the functions at our disposal are properly chosen, the deflection will not sensibly alter when for the distribution in question we substitute that which actually holds.

The first of the two states of strain referred to is that assumed by Kirchhoff as the basis of his first memoir on the subject, and afterwards employed by Thomson and Tait. It is defined by the properties (i.) that the middle surface is unextended, (ii.) that the particles originally in a line normal to this surface still form a straight line normal to it in the strained condition, and (iii.) that there is no traction across planes parallel to the middle surface.

Taking, as usual, the axes of x, y in the middle surface, in the unstrained condition, let w denote the deflection at any point (x, y) of this surface. In the state of strain above defined, the coordinates of any point of the substance are changed from (x, y, z) to (x', y', z'), where

$$x' = x - \frac{dw}{dx} z, \quad y' = y - \frac{dw}{dy} z,$$

so that the projections of a linear element $(dx, dy, 0)$ become

$$dx' = \left(1 - \frac{d^2w}{dx^2} z\right) dx - \frac{d^2w}{dx\,dy} z \,.\, dy,$$

$$dy' = -\frac{d^2w}{dx\,dy} z \,.\, dx + \left(1 - \frac{d^2w}{dy^2} z\right) dy.$$

We have here neglected the square of the angle which the normal to the middle surface makes with its original direction. Hence, for the elongations parallel to x, y, we have

$$e_{xx} = -\frac{d^2w}{dx^2} z, \quad e_{yy} = -\frac{d^2w}{dy^2} z,$$

while the shear in the plane xy is

$$e_{xy} = -2\frac{d^2w}{dx\,dy} z.$$

The elongation e_{zz} parallel to z is to be found from the condition that there is no traction across any plane parallel to the middle surface, that is

$$0 = p_{zz} = (\lambda + 2\mu)\, e_{zz} + \lambda\,(e_{xx} + e_{yy}),$$

the notation for the elastic constants being that introduced by Lamé, and now generally accepted by continental writers.* Substituting the value of e_{zz}, hence obtained, in the expressions for the remaining component tractions, we find

$$p_{xx} = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}(e_{xx}+\sigma e_{yy}),$$

$$p_{yy} = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}(e_{yy}+\sigma e_{xx}),$$

$$p_{xy} = \mu e_{xy},$$

$$p_{zz} = p_{yz} = 0,$$

where $\sigma, = \lambda/2(\lambda+\mu)$, is Poisson's ratio.

The applied forces necessary to maintain this state of strain are, per unit volume,

$$(X) = -\left(\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\right)$$

$$= \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}.z\frac{d}{dx}\nabla^2 w,$$

$$(Y) = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}.z\frac{d}{dy}\nabla^2 w,$$

$$(Z) = 0,$$

where $$\nabla^2 = d^2/dx^2+d^2/dy^2.$$

Replacing these by their force- and couple-resultants $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and

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* A few words on this difficult question of notation. As regards the elastic constants, a special symbol for the rigidity is in any case required, whilst it is comparatively unimportant in what manner the second elastic constant is defined. So many combinations of the two constants are needed—*e.g.*, the cubical compressibility, Young's modulus, Poisson's ratio, &c.—that it is impossible to devise a notation which shall give simple expressions to them all. Lamé's notation is convenient, and has found wider acceptance than any other. That of Thomson and Tait is in itself very neat, but is open to the objection that the letters m, n are so often wanted in other senses.

With respect to strains and stresses, some form of double suffix notation appears to the writer to be for many purposes almost indispensable. Such forms have at all events the advantage that they define themselves, and so give no trouble to readers who are accustomed to other notations. F. Neumann's symbols for the component stresses (X_x, X_y, X_z, &c.) are very expressive, but the corresponding notation devised by Kirchhoff for the strains (x_x, x_y, x_z, &c.) seems far less happy.

$\overline{L}$, $\overline{M}$, $\overline{N}$, reckoned per unit area of the plate, we find

$$\overline{X} = \int_{-h}^{h} (X)\, dz = 0, \quad \overline{Y} = 0, \quad \overline{Z} = 0,$$

$$\overline{L} = -\int_{-h}^{h} z\,(Y)\, dz = -A \frac{d}{dy} \nabla^2 w,$$

$$\overline{M} = \int_{-h}^{h} z\,(X)\, dz = A \frac{d}{dx} \nabla^2 w,$$

$$\overline{N} = 0,$$

where $2h$ is the thickness, and

$$A = \frac{8}{3} \frac{(\lambda+\mu)\,\mu}{\lambda+2\mu} h^3.$$

Next, for the stresses across normal sections of the plate. The stresses across a section perpendicular to x reduce to a couple whose components are, per unit length,

$$G_{xx} = -\int_{-h}^{h} p_{xy} z\, dz = (1-\sigma) A \frac{d^2 w}{dx\, dy}, \quad \text{about } Ox,$$

and

$$G_{xy} = \int_{-h}^{h} p_{xx} z\, dz = -A \left(\frac{d^2 w}{dx^2} + \sigma \frac{d^2 w}{dy^2} \right), \quad \text{about } Oy.$$

In like manner the stresses across a section perpendicular to y reduce to a couple whose components are

$$G_{yx} = -\int_{-h}^{h} p_{yy} z\, dz = A \left(\frac{d^2 w}{dy^2} + \sigma \frac{d^2 w}{dx^2} \right), \quad \text{about } Ox,$$

and

$$G_{yy} = \int_{-h}^{h} p_{xy} z\, dz = -(1-\sigma) A \frac{d^2 w}{dx\, dy}, \quad \text{about } Oy.*$$

We notice that

$$\frac{dG_{xx}}{dx} + \frac{dG_{yx}}{dy} = -\overline{L},$$

$$\frac{dG_{xy}}{dx} + \frac{dG_{yy}}{dy} = -\overline{M},$$

as evidently ought to be the case, since the stresses on the boundary of any portion of an elastic solid must balance the applied forces on

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* In the notation of Thomson and Tait,

$$G_{xx} = -G_{yy} = \Pi,$$
$$G_{xy} = -K, \quad G_{yx} = \Lambda.$$

the interior, the portion considered in the present instance being the element of the plate corresponding to the rectangular element $dx\,dy$ of the middle surface.

We shall require, presently, expressions for the stress-couples across *any* normal section. Let the normal to the plane of the section make an angle ϕ with the axis of x, and let the component couples about this normal and about the line in which the plane of the section meets the middle surface be $\mathfrak{N}$ and $\mathfrak{T}$, respectively. Considering the equilibrium of a triangular element of the plate bounded by this section, and by sections perpendicular to x and y, respectively, we find, taking moments about Ox, Oy,

$$G_{xx}\cos\phi + G_{yx}\sin\phi = \mathfrak{N}\cos\phi - \mathfrak{T}\sin\phi,$$

$$G_{xy}\cos\phi + G_{yy}\sin\phi = \mathfrak{N}\sin\phi + \mathfrak{T}\cos\phi,$$

whence

$$\mathfrak{N} = (1-\sigma)\,A\,(\cos^2\phi - \sin^2\phi)\,\frac{d^2w}{dx\,dy} + (1-\sigma)\,A\cos\phi\sin\phi\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right),$$

$$\mathfrak{T} = -\sigma A\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right)$$

$$-(1-\sigma)\,A\left(\frac{d^2w}{dx^2}\cos^2\phi + 2\frac{d^2w}{dx\,dy}\cos\phi\sin\phi + \frac{d^2w}{dy^2}\sin^2\phi\right).$$

3. The second state of strain which we shall consider is as follows. Keeping the middle surface fixed, let the particles on either side undergo displacements relative to it whose components are of the forms

$$\alpha = P\left(z - \frac{z^3}{3h^2}\right), \quad \beta = Q\left(z - \frac{z^3}{3h^2}\right), \quad \gamma = 0,$$

where P, Q are functions of x and y as yet undetermined. We will further suppose that the variations of P, Q, when x and y receive increments of the order h, may be neglected in comparison with the values of these functions themselves. This is to be regarded merely as a provisional assumption which may be justified *à posteriori* in any particular problem. On this supposition we may write, for the components of stress,

$$p_{xx} = 0, \quad p_{yy} = 0, \quad p_{zz} = 0,$$

$$p_{yz} = \mu Q\left(1 - \frac{z^2}{h^2}\right), \quad p_{zx} = \mu P\left(1 - \frac{z^2}{h^2}\right), \quad p_{xy} = 0;$$

and we find that the stresses across an element of a normal section

perpendicular to x reduce practically to a force parallel to z, whose value per unit length is

$$P_{zz} = \int_{-h}^{h} \mu P \left(1 - \frac{z^2}{h^2}\right) dz, = \tfrac{4}{3}\mu P h.$$

Similarly, the action across an element of a normal section perpendicular to y reduces to a shearing force

$$P_{yz} = \tfrac{4}{3}\mu Q h,$$

per unit length. Hence the above formulæ for α, β, γ may be written

$$\alpha = \frac{3}{4} \cdot \frac{P_{zz}}{\mu h}\left(z - \frac{z^3}{3h^2}\right), \quad \beta = \frac{3}{4} \cdot \frac{P_{yz}}{\mu h}\left(z - \frac{z^3}{3h^2}\right), \quad \gamma = 0.$$

By considering a triangular element of the plate as before, and resolving parallel to z, we find that the shearing force $\mathfrak{Z}$ across a section the normal to which makes an angle ϕ with Ox is, per unit length,

$$\mathfrak{Z} = P_{zz}\cos\phi + P_{yz}\sin\phi.$$

The shearing forces on the whole contour of an element $dx\,dy$ reduce to a force

$$\left(\frac{dP_{zz}}{dx} + \frac{dP_{yz}}{dy}\right) dx\,dy,$$

normal to the plate, and to a couple whose components are

$$P_{yz}\,dx\,.\,dy, \quad -P_{zz}\,dy\,.\,dx, \quad 0.$$

Since the values of α, β have been chosen so as to give zero stress on the two faces, this force and couple represent the stresses on the whole boundary of an element of the plate. Hence, if we reverse the signs, we get the force- and couple-resultants of the system of applied forces which must act on the interior of the element in order to maintain the strain in question.

4. Combining our results, we learn that the strain resulting from the superposition of the two states described in §§ 2 and 3, respectively, requires for its maintenance a certain distribution of applied force which is equivalent, for the element $dx\,dy$, to a force whose components are

$$0, \quad 0, \quad -\left(\frac{dP_{zz}}{dx} + \frac{dP_{yz}}{dy}\right) dx\,dy,$$

and a couple $\quad (\overline{L} - P_{yz})\,dx\,dy, \quad (\overline{M} + P_{zz})\,dx\,dy, \quad 0.$

In virtue of our fundamental assumption, no sensible change in the general form of the middle surface, or in the resultant stress across an element of a normal section, will ensue when for this distribution we substitute that which actually obtains, provided

$$\frac{dP_{xx}}{dx} + \frac{dP_{yx}}{dy} = -Z,$$

$$\overline{L} - P_{yx} = L,$$

$$\overline{M} + P_{xx} = M,$$

where Z represents the applied force normal to the plate, and L, M the components of applied couples about Ox, Oy, per unit area of the middle surface, in the sense explained in § 1.*

The last two equations give

$$P_{xx} = -A\frac{d}{dx}\nabla^2 w + M,$$

$$P_{yx} = -A\frac{d}{dy}\nabla^2 w - L,$$

and, substituting in the former, we obtain†

$$A\nabla^4 w = Z + \frac{dM}{dx} - \frac{dL}{dy},$$

the usual differential equation.

We have also, for the shearing force across any normal section,

$$\begin{aligned}\mathfrak{Z} &= P_{xx}\cos\phi + P_{yx}\sin\phi \\ &= -A\left\{\frac{d}{dx}\nabla^2 w \,.\cos\phi + \frac{d}{dy}\nabla^2 w \,.\sin\phi\right\} + M\cos\phi - L\sin\phi.\end{aligned}$$

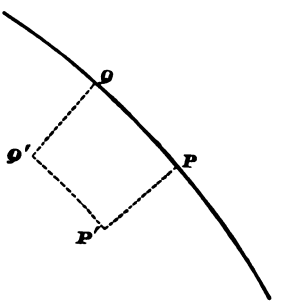
5. We come now to the boundary conditions. The difficulty here is, that in the immediate neighbourhood of the edge the mean state of strain is no longer represented, even approximately, by the assump-

* The most general specification of applied force would consist of forces X, Y, Z and couples L, M, 0; but the group Z, L, M acts quite independently of the group X, Y, the former producing curvature of the plate without extension of the middle surface, the latter extension without curvature. As we are here concerned only with the theory of flexure, we suppose X, Y to be zero. A like restriction is made as to the applied forces on the boundary.

† It is here assumed that L, M are continuous functions of x and y; Z, on the other hand, may be discontinuous. It is to be remembered that these functions merely express, as it were, the *average* distribution of applied force in the neighbourhood of the point (x, y).

tions of §§ 2, 3. In particular, a line of particles originally normal to the middle surface will not, in general, be normal to it after the deformation; it will not, in general, even remain straight. On the other hand, this exceptional state of things is practically confined to a narrow zone of the plate whose breadth is only a very moderate multiple of the thickness; beyond this region the statements in question apply, in the sense already explained.*

Now let PQ be an element ds of the boundary, and draw parallel to it, at a sufficient distance to fall outside the zone referred to, a line $P'Q'$, and let PP', QQ' be at right angles to the edge. The boundary conditions are obtained by considering the equilibrium of the element $PQQ'P'$ of the plate. Across $P'Q'$ we have a shearing force $\mathfrak{Z}\,ds$, and couples $\mathfrak{N}\,ds$, $\mathfrak{T}\,ds$, whose values are given by the formulæ of § 2, in which φ now denotes the angle which the normal to the edge makes with the axis of x. At the edge PQ we have the external forces, which may be specified by a force $\mathfrak{Z}_0\,ds$ parallel to z, and couples $\mathfrak{N}_0\,ds$, $\mathfrak{T}_0\,ds$ whose axes are the normal and tangent to the edge respectively. Across PP' (and this appears to be the key to the whole matter) we have a shearing force ζ which, unlike the shearing forces in the body of the plate, is specified not by its amount per unit length, but by its integral over a line extending inwards from the edge. This integral is finite in spite of the infinitely short length PP' over which it is taken.† The



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* See Thomson and Tait, §§ 727–9.

† This point, which does not appear to have been hitherto expressly noted, is illustrated by the problem discussed in Thomson and Tait, *loc. cit.* It is there shown that, if a certain distribution of force, constituting a uniform couple $\mathfrak{N}_0$ per unit length, be applied to the plane edge ($y=0$) of an otherwise infinite plate bounded by the planes $z=\pm h$, the component displacements α, β, γ at any point (x, y, z) are given by

$$\alpha = 3\,\frac{\mathfrak{N}_0}{\mu h}\left(\frac{2}{\pi}\right)^3 \Sigma\,\frac{(-1)^n}{(2n+1)^3}\,e^{-(2n+1)\pi y/2h}\sin\frac{(2n+1)\pi z}{2h},$$

$$\beta = 0, \quad \gamma = 0;$$

the plate being supposed to lie on the positive side of the plane zx. (The notation has been modified to correspond to that of this paper.) Hence

$$\zeta = \int_0^\eta \int_{-h}^{h} \mu\,\frac{d\alpha}{dz}\,dx\,dz = \frac{96}{\pi^4}\,\mathfrak{N}_0 \cdot \Sigma\,\frac{1-e^{-(2n+1)\pi\eta/2h}}{(2n+1)^4},$$

where η stands for the breadth PP' in the above figure. Since $e^{-10} = \cdot0000454$, we see that, if η exceeds three or four times the thickness, the series practically reduces to $\Sigma\,(2n+1)^{-4}$, or $\pi^4/96$, so that

$$\zeta = \mathfrak{N}_0,$$

as the *argument in the text*, applied to this particular case, would indicate.

corresponding shearing force across QQ' is $\zeta + d\zeta/ds \,.\, ds$. Hence, resolving parallel to z,

$$\mathfrak{Z}_0 - \mathfrak{Z} + \frac{d\zeta}{ds} = 0,$$

and taking moments about the tangent and normal, respectively,

$$\mathfrak{T}_0 - \mathfrak{T} = 0,$$

$$\mathfrak{N}_0 - \mathfrak{N} + \zeta = 0.$$

If we substitute the values of $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{Z}$, given in §§ 2, 4, in the equations which remain after elimination of ζ, viz., in

$$\mathfrak{T} = \mathfrak{T}_0,$$

$$\mathfrak{Z} - \frac{d\mathfrak{N}}{ds} = \mathfrak{Z}_0 - \frac{d\mathfrak{N}_0}{ds},$$

we obtain Kirchhoff's boundary conditions.

It was pointed out by Thomson and Tait that, if $\mathfrak{Z}_0$ and $d\mathfrak{N}_0/ds$ be increased by equal amounts, the state of things beyond the immediate neighbourhood of the edge is unaltered; and this remark forms the basis of their treatment of the boundary conditions.* We now see that the effect is on the distribution of the shearing force which we have denoted by ζ; viz., if $\phi'(s)$ be the common increment of the above functions, then ζ is increased by $-\phi(s)$.

The precise character of the boundary conditions laid down by Poisson and Kirchhoff respectively has often been discussed, and there is, perhaps, even yet room for comment on the subject. It is certainly remarkable, as a point of mathematical history, that the exceptional state of strain near the edge, on which, as we have seen, so much depends, should not have been recognized by either writer.

In the work of Poisson the fundamental assumption is, that when the various functions expressing the strain are expanded in powers of z, the distance from the middle surface, all terms after the first two may be neglected; and it is shown that, in the absence of stress on the two surfaces of the plate, the state of strain will then everywhere conform to the description given above in § 2. If this state is to hold right up to the edge, the three boundary conditions given by Poisson, viz. (in our notation)

$$\mathfrak{Z} = \mathfrak{Z}_0, \quad \mathfrak{T} = \mathfrak{T}_0, \quad \mathfrak{N} = \mathfrak{N}_0,$$

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* [*April*, 1890. The argument is presented in a very clear form by Boussinesq in his book entitled *Application des Potentiels à l'Étude de l'Équilibre . . . . des Solides Élastiques*, §§ 71, 72.]

are certainly necessary. It has been proved, however, by Kirchhoff that it is, in general, impossible to satisfy three independent boundary conditions. We infer that, with arbitrary values of $\mathfrak{Z}_0$, $\mathfrak{T}_0$, $\mathfrak{N}_0$, there will, in general, be the peculiar state of strain near the edge, and the accompanying distribution of the shearing force ζ, above described. That Poisson's assumption may be here violated is well shown in the case worked out by Thomson and Tait, to which reference has already been made, where the variation of strain through the thickness is given by circular functions of $\pi z/2h$.

Kirchhoff, in his original investigation, begins by postulating the state of strain of § 2, and proceeds to calculate the potential energy in terms of the curvature of the middle surface. He then applies the energy condition of equilibrium, equating the increment of potential energy in any slight change of deformation to the work done by the applied forces. In this process the internal forces, including the shearing force ζ, do not, of course, explicitly appear; but to make the argument accord with our present knowledge it would be necessary to recognize the existence of a narrow zone along the edge to which the fundamental postulates do not apply, and to assume that the potential energy of this zone may be neglected in comparison with that of the rest of the plate.

6. A simple but very instructive illustration of the point to which special attention has been directed above is furnished by a problem discussed by Thomson and Tait (§ 656) from a somewhat different point of view. Taking the case of a rectangular plate whose edges are parallel to x and y, let us ascertain what is the simplest* system of applied forces which will produce the uniform anticlastic curvature

$$w = Cxy.$$

It appears at once that we may suppose P_{xx}, P_{yy}, $\mathfrak{Z}$, Z, L, M to be zero. Again, at an edge perpendicular to x, we have

$$\mathfrak{N} = (1-\sigma)\,AC, \quad \mathfrak{T} = 0,$$

whilst at an edge perpendicular to y

$$\mathfrak{N} = -(1-\sigma)\,AC, \quad \mathfrak{T} = 0;$$

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* The problem, to find a system of forces which will produce a given deformation, is of course indeterminate, the only conditions being that

$$Z + \frac{dM}{dx} - \frac{dL}{dy}, \quad \mathfrak{T}_0, \quad \mathfrak{Z}_0 - \frac{d\mathfrak{N}_0}{ds}$$

are to have prescribed values.

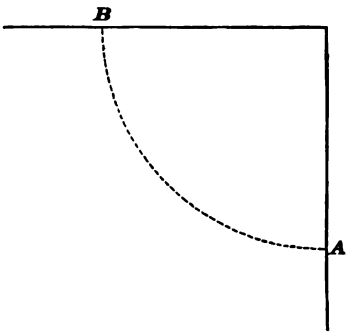
so that the boundary conditions are satisfied by $\mathfrak{T}_0 = 0$, $\mathfrak{N}_0 = 0$, $\mathfrak{Z}_0 = 0$, except at the actual corners. These points must be excepted on account of the term $d\mathfrak{N}/ds$ in the second condition, which is there infinite. If we suppose the corners to be rounded off, and integrate the condition in question over an infinitely short length of the edge including a corner, we find

$$-[\mathfrak{N}] = \int \mathfrak{Z}_0 ds,$$

the square brackets being used to indicate that the difference of the values of $\mathfrak{N}$ at the two limits is to be taken. For one corner these limits will correspond to $\phi = 0$, $\phi = \pi/2$; for a second to $\phi = \pi/2$, $\phi = 3\pi/2$; and so on. Hence

$$\int Z_0 ds = \pm 2\,(1-\sigma)\,AC;$$

so that we have the case of a plate bent by two contrary pairs of equal forces at the extremities of the two diagonals. If we consider the equilibrium of a portion of the plate cut off by a line AB drawn as in the figure, the applied force at the corner is balanced by the shearing forces which we have denoted by ζ $[= (1-\sigma)\,AC]$ acting at the points A, B, and by the flexural couples across the dotted line.



7. The number of problems which have been solved in the present subject is so small that I may be allowed to append a discussion of one or two very simple questions.

In the first place, let us take the case of waves propagated along an infinitely long band bounded by straight parallel edges $y = \pm b$, there being no external forces. The boundary conditions then reduce to

$$\frac{d}{dy}\left\{\frac{d^2w}{dy^2} + (2-\sigma)\frac{d^2w}{dx^2}\right\} = 0,$$

$$\frac{d^2w}{dy^2} + \sigma\frac{d^2w}{dx^2} = 0.$$

The general differential equation is, by d'Alembert's principle,

$$\frac{d^2w}{dt^2} + c^4\nabla^4 w = 0,$$

where $c^4 = A/\rho$, ρ being the mass per unit area. If we assume

$$w = Qe^{ik(x-vt)},$$

where Q is a function of y only, this gives

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 Q = \frac{k^2v^2}{c^4} Q.$$

The solution of this, appropriate to our present purpose, is

$$Q = C_1 \cosh m_1 y + C_2 \cosh m_2 y,$$

where m_1, m_2 are determined by

$$m_1^2 = k^2 + \frac{kv}{c^2}, \quad m_2^2 = k^2 - \frac{kv}{c^2}.$$

The boundary conditions then give

$$C_1 m_1 \{m_1^2 - (2-\sigma)k^2\} \sinh m_1 b + C_2 m_2 \{m_2^2 - (2-\sigma)k^2\} \sinh m_2 b = 0,$$

$$C_1 (m_1^2 - \sigma k^2) \cosh m_1 b + C_2 (m_2^2 - \sigma k^2) \cosh m_2 b = 0,$$

whence, remembering that $m_1^2 + m_2^2 = 2k^2$, we find

$$m_1 (m_2^2 - \sigma k^2)^2 \tanh m_1 b = m_2 (m_1^2 - \sigma k^2)^2 \tanh m_2 b,$$

the equation to determine v, the velocity of propagation.

If $m_1 b$, $m_2 b$ are small, this reduces to

$$m_1^2 (m_2^2 - \sigma k^2)^2 = m_2^2 (m_1^2 - \sigma k^2)^2,$$

whence we easily find $v = (1-\sigma^2)^{\frac{1}{2}} . kc^2$.

This coincides with the velocity of propagation of waves of length $2\pi/k$ along a uniform straight bar, as might have been anticipated from the fact that we have virtually assumed the wave-length to be great in comparison with the width of the band. Continuing the approximation a stage further, we find

$$v^2/k^2c^4 = 1 - \sigma^2 + \tfrac{2}{3}(1-\sigma)\sigma^2 k^2 b^2.$$

If, on the other hand, $m_1 b$, $m_2 b$ are moderately large, we may put

$$\tanh m_1 b = 1, \quad \tanh m_2 b = 1,$$

so that $$m_1 (m_2^2 - \sigma k^2)^2 = m_2 (m_1^2 - \sigma k^2)^2.$$

Rearranging, dividing by $m_1 - m_2$, and putting

$$m_1^2 + m_2^2 = 2k^2,$$

we get $$m_1^2 m_2^2 + 2(1-\sigma) k^2 m_1 m_2 - \sigma^2 k^4 = 0,$$

a quadratic to determine $m_1 m_2$. Since we have virtually taken m_1, m_2 to be positive, the positive root of this is to be chosen, and the wave-velocity is then given by

$$v^2/k^2c^4 = 1-m_1^2m_2^2/k^4$$
$$= (1-\sigma)\{3\sigma-1+2(1-2\sigma+2\sigma^2)^{\frac{1}{2}}\}.$$

The following table gives, in the second column, the values of v/kc^2 —*i.e.*, the ratio of the velocity of propagation to that of waves of the same length in an infinite plate—for various values of σ. The fourth column enables us to estimate the ratio which the breadth of the

σ	v/kc^2	m_1/k	m_2/k
0	1·000000	1·4142	·0000
·1	·999984	1·4142	·0040
·2	·999697	1·4141	·0174
·3	·998102	1·4135	·0436
·4	·992639	1·4116	·0858
·5	·978318	1·4065	·1472

band must bear to the wave-length in order that the above approximation may be valid. Thus, since $\tanh\pi$ falls short of unity by about 1/150000, we see that for $\sigma = \frac{1}{2}$ the approximation is ample so soon as the breadth exceeds three or four times the wave-length. For smaller values of σ much higher values of the ratio may be required. It is to be noticed that, however short the waves may be in comparison with the breadth, the velocity of propagation (except in the extreme case $\sigma = 0$) falls below its value for an unlimited plate by a finite, though not a very great, amount. The circumstances are, in fact, different; owing to the occurrence of the hyperbolic functions of $m_1 y$, $m_2 y$ in the value of w, the amplitude increases as we pass outwards from the centre of the band towards either edge, instead of remaining uniform. Thus the ratio of the amplitudes at the edge and at the centre is

$$\frac{C_1\cosh m_1 b + C_2\cosh m_2 b}{C_1+C_2},$$

which
$$= \frac{(m_1^2-m_2^2)\cosh m_1 b\,.\cosh m_2 b}{(m_1^2-\sigma k^2)\cosh m_1 b-(m_2^2-\sigma k^2)\cosh m_2 b}.$$

This is sensibly equal to unity in the case first considered, where $m_1 b$, $m_2 b$ were small; but when $m_1 b$, $m_2 b$ are considerable, and m_1

large in comparison with m_2, it is of the order $e^{m_2 b}$, and therefore large.* It appears that the waves propagated in the form of cylindrical corrugations, which are contemplated in the investigation for an infinite plate, form an ideal case which cannot be realized in a plate of finite dimensions, except in so far as the extensions (neglected in the above theory) which are called into play by the increase of amplitude towards the edges tend to keep this increase in check.

8. The vibrations of a rectangular plate have not yet been investigated in a rigorous manner. Part of the difficulty is in satisfying the conditions at the corners. It may be worth while to write down these conditions, as one conclusion of some interest can at once be drawn. Taking the origin at a corner, and the axes of x, y along the edges which meet there, we have, as before, for all points of the axis of x,

$$\frac{d}{dy}\left\{\frac{d^2w}{dy^2}+(2-\sigma)\frac{d^2w}{dx^2}\right\}=0,$$

$$\frac{d^2w}{dy^2}+\sigma\frac{d^2w}{dx^2}=0;$$

whilst, along the axis of y,

$$\frac{d}{dx}\left\{\frac{d^2w}{dx^2}+(2-\sigma)\frac{d^2w}{dy^2}\right\}=0,$$

$$\frac{d^2w}{dx^2}+\sigma\frac{d^2w}{dy^2}=0.$$

The further condition to be satisfied at the corner has already been obtained in § 6, viz., it is

$$[\mathfrak{M}]=-\int \mathfrak{B}_0 ds=0,$$

whence

$$\frac{d^2w}{dxdy}=0$$

at the origin. Since the first pair of boundary conditions may be differentiated with respect to x, and the second with respect to y, it easily follows that the differential coefficients of w of the second and

* The case, at which we have virtually arrived, of waves travelling along the straight edge of an otherwise unlimited plate may be treated more directly by taking the origin in this edge ($y = 0$), and assuming

$$w=(C_1e^{-m_1y}+C_2e^{-m_2y})e^{ik(x-ct)}.$$

The results coincide with those found above.

third orders all vanish at the origin. Hence near the corner the form of the plate will differ from a plane by small quantities of the *fourth* order. This may be compared with the state of things at the free end of a bar.

Appendix.

In the paper as originally drafted, there followed a discussion of one or two problems which can be worked out rigorously from the general equations of elasticity, and so serve in some measure as a test of the assumptions made in the theory of thin plates. The most important of these, viz., the propagation of waves in an infinite plate, has been fully treated by Lord Rayleigh in a paper* which has in the meantime appeared; I therefore confine myself to the following problem, which may serve as an illustration of the action of applied force:

An infinite plate, bounded by the planes $z = \pm h$, is subject to a force $C \cos kx$ per unit area of the middle surface, parallel to z; to find the deformation.

We assume for the component displacements

$$\alpha = \frac{d\phi}{dx} + \frac{d\psi}{dz}, \quad \beta = 0, \quad \gamma = \frac{d\phi}{dz} - \frac{d\psi}{dx} \quad \ldots\ldots\ldots\ldots (1),$$

where ϕ, ψ do not contain y. Let us first suppose that the external force is applied in the way of a bodily force of amount $C/2h \,.\, \cos kx$ per unit volume. The equations to be satisfied in the interior of the plate then are

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2\alpha &= 0, \\ (\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2\gamma + \frac{C}{2h}\cos kx &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots (2),$$

where

$$\delta = \frac{d\alpha}{dx} + \frac{d\gamma}{dz},$$

and ∇^2 now stands for $d^2/dx^2 + d^2/dz^2$. By differentiation we obtain

$$\left.\begin{aligned} \nabla^2\delta &= 0, \\ \mu\nabla^2\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + \frac{kC}{2h}\sin kx &= 0, \end{aligned}\right\}$$

* *Lond. Math. Soc. Proc.*, April, 1889.

or, if we assume that $\alpha \propto \sin kx$, $\gamma \propto \cos kx$,

$$\left.\begin{aligned} \left(\frac{d^2}{dz^2}-k^2\right)^2\phi &= 0, \\ \mu\left(\frac{d^2}{dz^2}-k^2\right)^2\psi + \frac{kC}{2h}\sin kx &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

A particular solution is

$$\left.\begin{aligned} \phi &= (F\sinh kz + Gkz\cosh kz)\cos kx, \\ \psi &= -\frac{C}{2\mu k^2 h}\sin kx + (H\cos kz + Kkz\sinh kz)\sin kx, \end{aligned}\right\} \ldots\ldots(4);$$

and on substitution we find that the original equations (2) are also satisfied, provided

$$(\lambda+2\mu)\,G = \mu K \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The boundary conditions are

$$p_{xx} = 0, \quad p_{xz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

or

$$\lambda\delta + 2\mu\frac{d\gamma}{dz} = 0, \quad \frac{d\alpha}{dz} + \frac{d\gamma}{dx} = 0,$$

for $z = \pm h$. The first of these gives, in conjunction with (5),

$$(F-H)\sinh kh + (G-K)\,kh\cosh kh = 0 \quad \ldots\ldots\ldots (7),$$

whilst the second leads to

$$(F-H)\cosh kh + (G-K)(\cosh kh + kh\sinh kh) + \frac{C}{4\mu k^2 h} = 0 \ldots (8).$$

Solving,

$$\left.\begin{aligned} F-H &= \frac{C}{4\mu k^3}\cdot\frac{\cosh kh}{\sinh kh\cosh kh - kh}, \\ G-K &= -\frac{C}{4\mu k^3}\cdot\frac{\sinh kh/kh}{\sinh kh\cosh kh - kh}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (9).$$

After a little reduction the consequent expressions for the displacements are found to be

$$\alpha = -\frac{C}{4\mu k}\left\{\cosh kh\,.\sinh kz - \frac{\sinh kh}{kh}\left(kz\cosh kz + \frac{\lambda+2\mu}{\lambda+\mu}\sinh kz\right)\right\}\sin kx$$
$$\div \{\sinh kh\cosh kh - kh\} \quad \ldots\ldots\ldots\ldots\ldots (10),$$

$$\gamma = \frac{C}{2\mu k^2 h}\cos kx$$
$$+\frac{C}{4\mu k}\left\{\cosh kh\,.\cosh kz - \frac{\sinh kh}{kh}\left(kz\sinh kz - \frac{\mu}{\lambda+\mu}\cosh kz\right)\right\}\cos kx$$
$$\div \{\sinh kh\cosh kh - kh\} \quad \ldots\ldots\ldots\ldots (11).$$

If kh be small—*i.e.*, if the variation of the applied force within a space compared with the thickness may be neglected—we find, to a first approximation,

$$\left.\begin{aligned} \alpha &= \frac{3}{8}\frac{\lambda+2\mu}{(\lambda+\mu)\mu}\cdot\frac{C}{k^3h^3}z\sin kx, \\ \gamma &= \frac{3}{8}\frac{\lambda+2\mu}{(\lambda+\mu)\mu}\cdot\frac{C}{k^4h^3}\cos kx, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12).$$

The latter formula agrees with the result given by the ordinary theory, viz.,

$$w = \frac{C}{k^4A}\cos kx,$$

where A has the value given in § 2. We further notice that

$$\alpha = -z\frac{d\gamma}{dx},$$

so that a line of particles originally straight and normal to the middle surface retains these properties after the flexure. This statement needs correction when we proceed to a second approximation; but, since α is an odd function of z, the curve into which such a line of particles is deformed has a point of inflexion where it crosses the middle surface; moreover, it is orthogonal at its extremities to the faces of the plate, since the shear in the plane zx there vanishes. It follows that the correction in question is very slight for moderate values of kh.

The formulæ for the component stresses give, on substitution from (10), (11),

$$p_{xx} = \lambda\delta + 2\mu\frac{d\alpha}{dx}$$

$$= \frac{C}{2}\left\{\frac{2\sinh kh}{kh}\sinh kz + \frac{\sinh kh}{kh}\,.\,kz\cosh kz - \cosh kh\,.\,\sinh kz\right\}\cos kx$$

$$\div \left\{\sinh kh\cosh kh - kh\right\} \quad \ldots\ldots\ldots\ldots (13),$$

$$p_{zz} = \mu\left(\frac{d\alpha}{dz} + \frac{d\gamma}{dx}\right)$$

$$= \frac{C}{2}\left\{\frac{\sinh kh/kh\,.(\cosh kz + kz\sinh kz) - \cosh kh\cosh kz}{\sinh kh\cosh kh - kh} - \frac{1}{kh}\right\}\sin kx$$

$$\ldots\ldots\ldots (14),$$

$$p_{xx} = \lambda\delta + 2\mu\frac{d\gamma}{dz}$$

$$= \frac{C}{2}\left\{\cosh kh \,.\, \sinh kz - \frac{\sinh kh}{kh}\,.\,kz\cosh kz\right\}\cos kx$$

$$\div \left\{\sinh kh\cosh kh - kh\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15).$$

For small values of kh these reduce to

$$\left.\begin{aligned} p_{xx} &= \frac{3}{2}\frac{C}{k^2h^2}\cdot\frac{z}{h}\cos kx = -4\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\,z\frac{d^2w}{dx^2}, \\ p_{zx} &= -\frac{3}{4}\frac{C}{kh}\left(1-\frac{z^2}{h^2}\right)\sin kx = -2\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\,.\,(h^2-z^2)\frac{d^2w}{dx^2}, \\ p_{zz} &= \frac{1}{4}C\frac{z}{h}\left(1-\frac{z^2}{h^2}\right)\cos kx = \frac{2}{3}\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\,.\,z\,(h^2-z^2)\frac{d^4w}{dx^4}, \end{aligned}\right\} \ldots (16).$$

The value of p_{xx} agrees with that obtained on the assumption of § 2; that of p_{zx} is small in comparison, whilst p_{zz} is of a still higher order of smallness. The second forms (in terms of w) are given on account of their greater generality; they hold for any distribution of bodily force which depends on x only,* since any such distribution can be expressed by Fourier's theorem in a series of terms of the forms $\cos kx$, $\sin kx$.

Let us next suppose that the external forces act on the two *faces* of the plate, and are equally divided between them, so that the surface conditions (6) are replaced by

$$\left.\begin{aligned} p_{zz} &= \tfrac{1}{2}C\cos kx, \quad \text{for } z = +h, \\ &= -\tfrac{1}{2}C\cos kx, \text{ for } z = -h, \\ p_{z} &= 0. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (17).$$

The equations to be satisfied in the interior now are

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2 a = 0, \\ (\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2\gamma = 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (18);$$

whence

$$\left(\frac{d^2}{dz^2} - k^2\right)^2\phi = 0, \quad \left(\frac{d^2}{dz^2} - k^2\right)^2\psi = 0.$$

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* Provided, of course, that its rate of variation within a space comparable with h may be neglected.

Assuming

$$\left.\begin{aligned}\phi &= (F\sinh kz + Gkz\cosh kz)\cos kx,\\ \psi &= (H\cosh kz + Kkz\sinh kz)\sin kx,\end{aligned}\right\}$$

we find, as before, that the equations (18) are satisfied, provided

$$(\lambda+2\mu)\,G = \mu K.$$

Taking account of this, the boundary conditions (17) give

$$\left.\begin{aligned}(F-H)\sinh kh + (G-K)\,kh\cosh kh &= \frac{C}{4\mu k^2},\\ (F-H)\cosh kh + (G-K)\,(\cosh kh + kh\sinh kh) &= 0,\end{aligned}\right\} \quad \ldots (19);$$

whence

$$\left.\begin{aligned}F-H &= \frac{C}{4\mu k^2}\cdot\frac{\cosh kh + kh\sinh kh}{\sinh kh\cosh kh - kh},\\ G-K &= -\frac{C}{4\mu k^2}\cdot\frac{\cosh kh}{\sinh kh\cosh kh - kh},\end{aligned}\right\} \quad \ldots\ldots\ldots (20).$$

The values of the displacements are found to be

$$\alpha = \frac{C}{4\mu k}\left\{\cosh kh\left(\frac{\mu}{\lambda+\mu}\sinh kz + kz\cosh kz\right) - kh\sinh kh\,.\sinh kz\right\}\sin kx$$
$$\div\left\{\sinh kh\cosh kh - kh\right\} \ldots\ldots\ldots\ldots (21),$$

$$\gamma = \frac{C}{4\mu k}\left\{\cosh kh\left(\frac{\lambda+2\mu}{\lambda+\mu}\cosh kz - kz\sinh kz\right) + kh\sinh kh\cosh kz\right\}\cos kx$$
$$\div\left\{\sinh kh\cosh kh - kh\right\} \ldots\ldots\ldots\ldots (22).$$

These may be compared with (10) and (11). When kh is small, the values of α and γ to a first approximation coincide with those given in (12). The stresses p_{xx}, p_{xz} are also found on examination to retain the same approximate values (16) as before, whilst for p_{zz} I find

$$p_{zz} = \frac{3}{4}\,C\,.\,\frac{z}{h}\left(1-\frac{z^2}{3h^2}\right)\cos kx$$
$$= \frac{2}{3}\,\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\,.\,z\,(3h^2-z^2)\,\frac{d^4w}{dx^4} \ldots\ldots\ldots\ldots (23).$$

We might also consider the case where the forces on the two faces are equal and opposite, so that the boundary conditions are

$$p_{xz} = \frac{C}{2}\cos kx, \quad p_{zz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24),$$

for $z = \pm h$. We should find

$$\left.\begin{aligned} \phi &= (F\cosh kz + G\,.\,kz\sinh kz)\cos kx, \\ \psi &= (H\sinh kz + K\,.\,kz\cosh kz)\sin kx, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (25),$$

where, as before,

$$(\lambda+2\mu)\,G = \mu K;$$

and the rest of the solution is obtained by simply interchanging the functions cosh kh, sinh kh, in equations (19). Hence

$$\left.\begin{aligned} F-H &= \frac{C}{4\mu k^3}\cdot\frac{\sinh kh + kh\cosh kh}{\sinh kh\cosh kh + kh}, \\ G-K &= -\frac{C}{4\mu k^3}\cdot\frac{\sinh kh}{\sinh kh\cosh kh + kh}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (26).$$

When kh is small, these lead to

$$\alpha = -\frac{C}{8k}\cdot\frac{\lambda}{(\lambda+\mu)\,\mu}\cdot\sin kx, \quad \gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (27),$$

at the middle surface, as might have been found by a much more elementary process. The effect of the given distribution is to produce an extension of the plate, but the displacement α is of the order k^3h^3 as compared with the value (12) of γ in the former case.

By superposition we can obtain from this case and the preceding the exact solution of the problem when the applied force is divided in any constant ratio between the two faces, but the resulting deformation is seen to be practically the same as in the case of an equal division.

The fundamental assumption of § 1 may be examined by taking kh large. In the case of a bodily force the formula (11) makes

$$\gamma = \frac{C}{2\mu k^3}\cos kx,$$

nearly, at the middle surface; *i.e.*, it varies ultimately as the square of the wave-length. When the force is applied to the two faces, we learn from (22) that γ is of the order e^{-kh} at the middle surface; in fact, the stress is now confined to a thin superficial stratum on each side.

Thursday, January 9th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. J. E. Campbell, B.A., Fellow and Lecturer of Hertford College, and Lecturer at University College, Oxford, was elected a Member.

The following communications were made:—

On the Deformation of an Elastic Shell: Prof. H. Lamb.

On the relation between the Logical Theory of Classes and the Geometrical Theory of Points: Mr. A. B. Kempe.

On the Correlation of Two Spaces, each of Three Dimensions: Dr. T. A. Hirst.

Note on the Simultaneous Reduction of the Ternary Quadric and Cubic to the forms $Ax^3+By^3+Cz^3+Dw^3$, $ax^3+by^3+cz^3+dw^3$: The President.

The following presents were received:—

"Educational Times," for January.

"Nautical Almanac," for 1893, 8vo; London.

"The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S." Vol. II., 4to; Cambridge, 1889. Two copies.

"Problems on the Motion of Atoms," by J. K. Smythies, 4to; London, 1885.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 5.

"Bulletin des Sciences Mathématiques," Tome XIII., December, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 11.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 5 and 6.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 96.

"Atti del Reale Istituto Veneto," Tomo VII., Ser. VI., Disp. 3 to 10.

"Nieuw Archief voor Wiskunde," Deel XVI., Stuk 2.

"Prace Matematyczno-Fizyczne," Tom. II., Zeszyt 1.

"Festschrift, herausgegeben von der Mathematischen Gesellschaft in Hamburg anlässlich ihres 200 jährigen Jubelfestes, 1890," erster Teil.

Ditto, ditto, Sonder-Abzug.

"Vierde Rapport van de Huygens-Commissie," 8vo; Amsterdam, 1889.

On the Correlation of Two Spaces, each of Three Dimensions.

By Dr. T. Archer Hirst, F.R.S.

[*Read Jan. 9th*, 1890.]

Introductory.

1. On November 12th, 1874, when resigning to my successor the Chair of President of the London Mathematical Society, I had occasion to exhibit, and to comment upon the following three, now well-known, formulæ for determining the number of ways in which two spaces, each of three dimensions, may be correlated, in the most general manner, so as to satisfy fifteen* given conditions, or, as we may more briefly say, for finding the *number of correlations* which satisfy any fifteen single conditions:—

$$4\mu = \pi + 3\omega + 2\psi,$$
$$4\rho = 3\pi + \omega + 2\psi,$$
$$2\nu = \pi + \omega + 2\psi.\dagger$$

2. In these formulæ, μ, ρ, ν denote, respectively, the number of correlations which, besides satisfying fourteen of the given conditions, likewise render a pair of given points, planes, or lines (one in each space) conjugate to one another—in other words, so related that the plane, point, or line corresponding to either of the two given elements shall be incident with the other.‡

3. The remaining symbols in the three formulæ have reference to the number of exceptional correlations which satisfy the fourteen conditions above referred to. The number of *central* correlations which do so is denoted by π; while ω and ψ indicate, respectively, the number of *planar*, and the number of *axial* ones which possess the same property.

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* Compare Chasles, *Aperçu Historique*, Seconde Edition, 1875, p. 766, Art. 293.

† *Proceedings of the London Mathematical Society*, Vol. VI., p. 7.

‡ A point and plane in the same space are said to be incident when the former lies in the latter; two lines are incident, of course, when they have a point in common, and, therefore, lie in one and the same plane.

4. The terms "central," "planar," and "axial" were, and are here again, applied to those exceptional or *special* forms of correlation whose analogues, in the ordinary theory of reciprocal polars relative to a quadric surface, present themselves whenever that surface degenerates either to a *cone*, a *conic*, or a *pair of planes*,—supplemented, it must be remembered, in the last case, by a *pair of points* situated on the intersection of those planes.

5. In a *central* correlation, in fact, there is a fixed singular point, or centre, in each space whose corresponding plane in the correlated space is perfectly arbitrary. Through this centre, moreover, passes the plane which corresponds to every non-central point of the other space, and this plane remains fixed, no matter how the point in question may be moved along one and the same line passing through the centre.

To the line last mentioned corresponds, strictly speaking, any line whatever situated in the plane corresponding to each of its points.

6. In a *planar* correlation, on the other hand, there is a fixed singular plane, in each space, whose corresponding point, in the other space, is indeterminate. In each such plane, however, every point lies which corresponds to a plane not coincident with the singular one of the correlated space, and that point remains fixed, no matter how the plane may be turned around its intersection with the singular one.

To the intersection just referred to, regarded as a given line, corresponds, in reality, any line whatever passing through the above-mentioned point.

7. In an *axial* correlation, lastly, there is a fixed line (axis) in each space through which all planes pass that correspond to points in the correlated space not situated on the axis thereof, and in which all points lie that correspond to planes not passing through that axis.

8. Moreover, each plane through an axis, and the plane through any point corresponding thereto, as well as through the associated axis, are corresponding elements of two projective pencils of planes; and, similarly, each point of an axis, and the intersection with the latter's associate of any plane corresponding to that point, are corresponding elements of two projective rows.

9. It may be added that, of two corresponding *lines*, in an axial correlation, each is necessarily incident with an axis at a determinate point and in a determinate plane—unless, indeed, one of the two lines were *coincident* with an axis, when the other would be perfectly indeterminate.

10. That fourteen conditions are necessary and sufficient to determine the singular elements of each of the above three special forms of correlation may be easily shown, directly, thus: In a *planar* correlation, six conditions determine the two associated singular planes, and eight the correlation between them.* Similar conclusions obviously apply to a *central* correlation; and, in an *axial* one, eight conditions determine the two associated axes, so far as position is concerned, three the projectivity of the two pencils of planes through these axes, and three, finally, that of the rows of points situated upon them.

11. By means of the three formulæ of Art. 1, it became possible, obviously, to reduce all problems concerning correlations which satisfy fifteen conditions to simpler ones having reference, solely, to special correlations of the three types above described, and satisfying fourteen, at most, of the given fifteen conditions.

12. But, although greatly interested in the solution of this problem, my duties connected with the Royal Naval College, Greenwich, precluded my undertaking it in 1875, and I felt myself impelled to solicit the co-operation of my friend Professor Sturm, then resident at Darmstadt, and now at Münster; he having, in fact, already treated the closely allied problem of Projectivity in Space†—the problem, in short, upon which the determination of axial correlations immediately depends. He at once consented to extend his researches to central and planar correlations, and, in due time, he published the elaborate memoir on these three subjects which appeared in the *Mathematische Annalen* (Vol. 12) during the year 1877, under the title, "*Ueber Correlative oder reciproke Bündel.*"

13. The continued pressure of other engagements, however, prevented any prompt application, even, of Professor Sturm's valuable results; and when, at length, I was able to resume my studies, I soon found that the whole subject was susceptible of still further simplification than the one I had submitted for my friend's consideration.

This at once resulted from the introduction of still more special forms of correlation than those above designated as central, planar, and axial,—forms, in short, upon which these latter themselves depend, in the same manner that a general correlation, as we have seen, depends upon the former.

14. In consequence of the detection of this further simplification,

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* *Proceedings of the London Mathematical Society*, Vol. v., pp. 40–70, 1874.

† *Mathematische Annalen*, 1873, Band vi., p. 513.

my original intention of immediately applying Sturm's results was abandoned, and the treatment exhibited in the present paper was adopted in its place.

Except for purposes of verification, therefore, I shall no longer have occasion to refer to the intrinsically valuable results of my friend's memoir of 1877.*

Higher, special forms of Correlation.

15. Before proceeding further, however, it becomes necessary to describe, briefly, the still more special forms of correlation referred to in Art. 13. For convenience, they may be referred to as special correlations of the second and third orders, those described in Arts. 5, 6, and 7 being of the first order.

16. Their nature will be best understood by once more recurring to the familiar theory of reciprocal polars relative to a quadric surface.

First. The *cone* into which this quadric surface has already been supposed to break up (Art. 4) may itself degenerate either into a *pair of planes* or into a *pair of right lines.*

Secondly. The *conic* into which the quadric surface, in Art. 4, has broken up, may further degenerate either to a *pair of right lines* or to a *pair of points.*

Thirdly. Of the pair of planes with a pair of points which represented the quadric surface in Art. 4, either the former or the latter may be coincident. In one case we have, once more, a *pair of points,* in a definite plane; in the other a *pair of planes,* through a definite point.

17. In short, in whichever of the above three ways the quadric surface may ultimately degenerate, we arrive at one or other of three well-determined special forms of correlation of the second order,—correlations, in short, which in the more general theory, having reference to two distinct spaces, may be appropriately defined as (1) *central-planar,* (2) *planar-axial,* or (3) *central-axial,* respectively.

18. In a central-planar correlation there is, in each space, a fixed centre and a fixed plane passing through it. In a planar-axial one, on the other hand, there is, in each space, a fixed plane and a fixed

* In this same year 1877, on my return from Egypt, where all the numerical results of the present paper were first obtained, I applied a precisely similar simplification to the simpler problem of the Correlation of Two Planes (see *Proc. Lond. Math. Soc.*, Vol. VIII., pp. 262–273.

line situated therein. In a central-axial correlation, finally, we have, in each space, an axis and a centre situated therein.

In all three cases the correspondence between the points, planes, and lines of the two correlated spaces partakes of the characters which distinguish each of the two special forms of correlation of which it is a combination, and which have been already sufficiently described in Arts. 5 to 9.

19. A special correlation of the first order, as already observed (Art. 10), is determined by fourteen conditions; one of the second order, however, requires *thirteen* only.

In a central-planar correlation, for instance, six of these would suffice to determine the two centres; four more the two singular planes passing through the latter; and three, finally, to establish the projectivity of the pencils situated in these planes, and around those centres.

In a planar-axial correlation, again, six conditions would determine the two associated singular planes; four more the axes in those planes; and three, as before, the projectivity of the points on the latter.

Lastly, in a central-axial correlation, eight conditions might be appropriated to the determination of the positions of the two associated axes; two more to those of their centres; and three, finally, to the establishment of the projectivity of the pencils of planes passing through those axes.

20. It should further be observed that in the twofold system of correlations which satisfy thirteen common (single) conditions, there will, in general, be a onefold system of special ones of the first order, and a finite number of those of the second.

Of the latter the number of those which are central-planar may be indicated by $(\pi\omega)$, the number of planar-axial ones by $(\omega\psi)$, and the number of central-axial correlations by $(\pi\psi)$—which symbols, of course, are always to be regarded as undecomposable.

21. But the process of degeneration described in Arts. 16 to 18 may, obviously, proceed one step further. A central-planar correlation may likewise become axial, a planar-axial one may acquire a centre, or a central-axial one a plane.

In each case we thus arrive at a special correlation of the third order, which may be termed *central-planar-axial*. It is characterised by having, in each of the two correlated spaces, a singular point (centre), a singular plane passing through that point, and a singular axis situated in this plane and likewise passing through that point.

22. Such a special correlation is determined, manifestly, by

twelve conditions. To determine the two associated centres, for instance, six would be required; four more might then be appropriated to the determination of the singular planes passing through those centres, and two would remain for the determination of the axes situated in these planes, and likewise passing through those centres.

23. In a threefold system of correlations satisfying twelve common (single) conditions (which would include a twofold system of degenerate ones of the first order, as well as a onefold system of special correlations of the second order), there will be, therefore, in general, a finite number, say $(\pi\omega\psi)$, of the above-defined special correlations of the third order.

Fundamental Formulæ.

24. I now proceed to the establishment of the three formulæ of Art. 1, or rather of the following three simpler ones, whereof the former three are mere algebraical transformations:—

$$\left.\begin{aligned} 2\mu &= \nu+\omega,^{*} \\ 2\rho &= \nu+\pi, \\ 2\nu &= \mu+\rho+\psi. \end{aligned}\right\}$$

25. To commence with the first of these, let a_1 and b_1 be any two lines in the correlated spaces A and B respectively, and let B_1' and B_1'' be any two fixed points in b_1. There are, by hypothesis (Art. 2), μ correlations of the onefold system under consideration, in which any assumed point A_1' in a_1 is conjugate to the fixed one B_1' in b_1. In each of these correlations, moreover, the plane corresponding to the second fixed point B_1'' in b_1 will cut a_1 in a point A_1'' conjugate to B_1''; so that in a_1 we have μ points A_1'' corresponding to each point A_1' of that line. Similarly, any one of these points A_1'' is conjugate to B_1'' in μ correlations of the system; and, relative to each of these, B_1' has a conjugate point A_1' situated in a_1. Between the pairs of points A_1' and A_1'' of this line, therefore, a (μ, μ) correspondence exists, and, assuming the latter to be algebraical, there must be 2μ united points, —that is to say, points, which in one and the same correlation

* This first of the three formulæ expresses, of course, the fact that in a onefold system of correlations satisfying fourteen given conditions, double the number of those in which two given points, one in each space, are conjugate to one another is equal to the number in which two given lines have that property, together with the number of planar correlations included in the system under consideration.

of the system are each, simultaneously, conjugate to B_1' as well as to B_1''.

26. Very little consideration is necessary, however, in order to recognize that there are two, and only two, kinds of correlation in the system wherein such united points present themselves, viz.: (1) the ν correlations, in which a_1 and b_1 are themselves conjugate lines, and (2) the ω planar correlations.

In the first case, the planes corresponding to B_1' and B_1'' intersect in the line corresponding to b_1. This line is incident with a_1, since, by hypothesis, a_1 and b_1 are conjugate to one another (Art. 2). The above-mentioned planes, therefore, cut a_1 in points which are not merely conjugate to B_1' and B_1'', respectively; they do so in points that are coincident with one another.

In the second case, to the perfectly arbitrary points B_1' and B_1'' correspond, in each planar correlation of the system, planes which are coincident with the singular plane of that system situated in the correlated space A. Here too, therefore, the points in a_1, conjugate to B_1' and B_1'', are coincident with one another.

27. The second of the three formulæ of Art. 24 follows at once from the first by the Principle of Duality.

28. The third is obtained readily by establishing the relation that must necessarily exist between the rays a_1' and a_1'' of an arbitrarily chosen plane-pencil in the space A which are, respectively, conjugate, in one and the same correlation of the system under consideration, with any two fixed incident lines b_1' and b_1'' in the space B. This correspondence is a (ν, ν) one; because, by hypothesis, there are ν correlations of the system, in which any ray a_1' (or a_1'') is conjugate with b_1' (or b_1''), and in each of these correlations the ray corresponding to b_1'' (or b_1') must be incident with, and thus determine, a ray a_1'' (or a_1').

29. A little reflection, however, suffices again to convince us that there are three, and only three, sources whence the 2ν united rays of the above correspondence can possibly proceed, viz.: (1) the μ correlations of the system in which the centre A_1 of the given pencil and the intersection B_1 of the two fixed lines b_1' and b'' are conjugate points; (2) the ρ correlations of the system in which the plane of the pencil in A is conjugate to that of the two fixed lines in B; and (3) the ψ axial correlations which are included in the system.

In the first of these three cases, in fact, the rays corresponding to

the fixed ones b_1' and b_1'' must lie on a plane passing through the point A_1, which is conjugate to their intersection; consequently, they must pierce the plane of the pencil in points collinear with the centre A_1 thereof.

In the second case, by the principle of duality, the lines corresponding to the fixed ones, in B, will pierce the plane of the pencil in A in one and the same point, and consequently again be incident with one and the same ray of that pencil.

In the third case, lastly, the fixed lines in B, being arbitrarily situated relative to the singular axis, in that space, of every axial correlation of the system, the lines in A, corresponding to them, will both coincide, necessarily, with the singular axis, in this space, of the correlation in question. The trace of this axis on the plane of the pencil around A_1 determines a ray of the latter, therefore, which is, once more, conjugate, in one and the same correlation of the system, to both the fixed lines b_1' and b_1''.

30. Formulæ similar to those of Art. 24, and established in a similar manner, connect what may be appropriately termed the characteristics and singularities of *systems* of special correlations of the first order satisfying thirteen common conditions.

The six results (three pairs) may be thus denoted:—

$$2\pi\mu = \pi\nu + (\pi\omega)$$
$$2\psi\mu = \psi\nu + (\omega\psi)$$

$$2\omega\rho = \omega\nu + (\pi\omega)$$
$$2\psi\rho = \psi\nu + (\pi\psi)$$

$$2\pi\nu = \pi\mu + \pi\rho + (\pi\psi)$$
$$2\omega\nu = \omega\mu + \omega\rho + (\omega\psi).$$

The first of these formulæ, for instance, expresses the fact that in a onefold system of central correlations, satisfying thirteen given conditions, double the number in which two given points, one in each space, are conjugate to one another, is equal to the number in which a given pair of lines are so, together with the number which, besides being central, are likewise planar correlations (Art. 18).

31. We may, indeed, proceed one step further by establishing the following three formulæ, which have reference to the systems o

special correlations, of the second order, that satisfy twelve common conditions:—

$$2(\pi\psi)\mu = (\pi\psi)\nu + (\pi\omega\psi)$$
$$2(\omega\psi)\rho = (\omega\psi)\nu + (\pi\omega\psi)$$
$$2(\pi\omega)\nu = (\pi\omega)\mu + (\pi\omega)\rho + (\pi\omega\psi).*$$

The last of these formulæ, it may be observed, expresses the fact that in a onefold system of central-planar correlations, satisfying twelve given conditions, double the number in which two given lines, one in each space, are conjugate, is equal to the number in which two given points are so, together with the number in which two given planes have the same relation, and to the number of correlations in the system which are not merely central and planar, but likewise axial (Art. 21).

32. In virtue of the three last formulæ of the six recorded in Art. 30, and of the three previously given in Art. 24, the number of ordinary correlations which satisfy any fifteen given elementary single conditions may now be gradually deduced from certain numerical results which are known *à priori*.

Central-Planar-Axial Correlations.

33. I commence with the direct determination of the number $(\pi\omega\psi)$ of central-planar-axial correlations which satisfy any twelve given elementary conditions.

This is equivalent, in fact, to finding the number of different ways in which the singular elements of such a correlation (Art. 21) may be determined so as to satisfy, in each of the correlated spaces, *six* conditions of incidence. For it is easy to see that, if two given points, planes, or lines, one in each space, are to be conjugate, relative to any central-planar-axial correlation, it is necessary, and sufficient, that one of the two associated singular planes, centres, or axes of that correlation should be incident with *one* of the given pair of conjugate elements. Its associate, so far as situation in its own space is concerned, is perfectly unconditioned by the *other* element of the given pair.

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* A comparison of the nine formulæ of this and the preceding article with the three given in Art. 24 cannot fail to remind the reader of that "symbolical multiplication" of which Schubert has made so many successful applications in his admirable "*Kalkul der abzählenden Geometrie*" (Leipzig: Verlag von B. G. Teubner, 1879). I have, in fact, as announced, by anticipation, on p. 195 of the latter work, applied the calculus in question to the present problem of correlation in space. In this paper, however, a recurrence to older, and more generally familiar, methods *has been* deemed desirable.

34. This fact, indeed, suggests the desirability, for the sake of clearness of exposition, of resolving the *complete* signature of the twelve conditions into two *partial* ones; indicative, respectively, of the nature of the six determining data in each of the two spaces.

Thus: if (l, m, n), where $l+m+n=12$, be the complete signature, indicating, respectively, the total number of given pairs of conjugate points, planes, and lines; and if (l', m', n') and (l'', m'', n''), where $l'+m'+n'=l''+m''+n''=6$, be the partial signatures into which the above complete one may be resolved; then l', m', n', for the one space, and l'', m'', n'', for the other, will indicate, respectively, the number of points, planes, and lines with which, alone, the singular elements of that space are supposed to be incident.

35. With respect to the limits between which the last-named six quantities must lie, in order to secure values of $(\pi\omega\psi)$ different from zero, it may be observed that, since no plane can be made to pass through more than three given points, and since no point can be made to lie in more than three given planes, none of the four quantities l', l'', m', m'' can exceed the number 3.

The fact, moreover, that no line can be made to meet, or be incident with, more than four given ones, at once gives 4 for the superior limit of each of the quantities n' and n''. Here, however, the circumstance that there are always *two* lines which are incident with four given ones has to be borne in mind.

36. The inferior limit of each of the six quantities l', m', n'; l'', m'', n'' is clearly 1; for, if any one of these six quantities vanished, not only would one of the three constituents of the singular elements, in one of the two spaces, be insufficiently determined, but the two others, therewith incident, would remain subject to six incompatible conditions.

The only five cases indeed, in which, at first sight, any doubt could arise on this point, are

$$(0, 3, 3),\quad (0, 2, 4),\quad (3, 0, 3),\quad (2, 0, 4),\text{ and }(3, 3, 0).$$

But the insufficiency of these data, on the one hand, and their incongruity, on the other, are too obvious to require further elucidation.

37. Thirteen cases alone demand detailed treatment; all others are deducible from them by the law of duality. These thirteen cases are collected in the next following table. In the *first* column of it their complete signatures (l, m, n) are given, arranged according to ascending values of n. In the *second* column the resolution, in all possible ways, is shown of each complete signature into two partial

ones. In the *third* column the values of $(\pi\omega\psi)$ answering to each such resolution is exhibited as a product of six factors.

38. Taken in order, the *first three* of these six factors give, respectively, the number of ways in which the points, the planes, and the lines, indicated by *either* of the two associated partial signatures, may be selected from the total number of these three elements exhibited in the complete signature. The *fourth* and *fifth* factors, each equal to 1 or 2, give, for the spaces to which the two partial signatures respectively refer, the number of axes which satisfy the given conditions of incidence; each such axis being supplemented, of course, by its appropriate plane and centre. The *sixth* factor, finally, which is likewise equal, numerically, either to 1 or 2, gives the frequency of the solution, above described; the latter is due, exclusively, to an interchange of the data between the two correlated spaces.

(l, m, n)	(l', m', n') (l'', m'', n'')	$(\pi\omega\psi)$
(6, 4, 2)	(3, 2, 1) (3, 2, 1)	20 . 6 . 2.1.1.1 = 240
(5, 5, 2)	(3, 2, 1) (2, 3, 1)	10 . 10 . 2.1.1.2 = 400
(6, 3, 3)	(3, 2, 1) (3, 1, 2)	20 . 3 . 3.1.1.2 = 360
(5, 4, 3)	(3, 2, 1) (2, 2, 2)	10 . 6 . 3.1.2.2 = 720
	(3, 1, 2) (2, 3, 1)	10 . 4 . 3.1.1.2 = 240
(6, 2, 4)	(3, 1, 2) (3, 1, 2)	20 . 2 . 6.1.1.1 = 240
(5, 3, 4)	(3, 2, 1) (2, 1, 3)	10 . 3 . 4.1.2.2 = 480
	(3, 1, 2) (2, 2, 2)	10 . 3 . 6.1.2.2 = 720
(4, 4, 4)	(3, 2, 1) (1, 2, 3)	4 . 6 . 4.1.2.2 = 384
	(3, 1, 2) (1, 3, 2)	4 . 4 . 6.1.1.2 = 192
	(2, 3, 1) (2, 1, 3)	6 . 4 . 4.1.2.2 = 384
	(2, 2, 2) (2, 2, 2)	6 . 6 . 6.2.2.1 = 864
(5, 2, 5)	(3, 1, 2) (2, 1, 3)	10 . 2 . 10.1.2.2 = 800
(4, 3, 5)	(3, 2, 1) (1, 1, 4)	4 . 3 . 5.1.2.2 = 240
	(3, 1, 2) (1, 2, 3)	4 . 3 . 10.1.2.2 = 480
	(2, 2, 2) (2, 1, 3)	6 . 3 . 10.2.2.2 = 1440
(4, 2, 6)	(3, 1, 2) (1, 1, 4)	4 . 2 . 15.1.2.2 = 480
	(2, 1, 3) (2, 1, 3)	6 . 2 . 20.2.2.1 = 960

$$(3,3,6)\begin{cases}(2,\ 2,\ 2)\ (1,\ 1,\ 4) & 3.\ 3.15.2.2.2 = 1080\\ (2,\ 1,\ 3)\ (1,\ 2,\ 3) & 3.\ 3.20.2.2.2 = 1440\end{cases}$$

$$(3,2,7)\quad (2,\ 1,\ 3)\ (1,\ 1,\ 4)\quad 3.\ 2.35.2.2.2 = 1680$$

$$(2,2,8)\quad (1,\ 1,\ 4)\ (1,\ 1,\ 4)\quad 2.\ 2.70.2.2.1 = 1120$$

39. From the above thirteen cases, nine others, as before observed, are at once deducible by the law of duality; for it is in virtue of this law that $(\pi\omega\psi)$ has the same numerical value for any two complete signatures, such as (l, m, n) and (m, l, n).

These, however, as well as the collected results of the preceding table, omitting all partial signatures, have been included in the following one, where, for economy of space, the expedient has been adopted of a double, superior and inferior, heading;—an expedient similar, in short, to that with which ordinary tables of logarithms have long familiarized us.

It may be here added that all values of $(\pi\omega\psi)$ vanish which answer to complete signatures other than those given in the following table:—

(l, m, n)	$(\pi\omega\psi)$
(6, 4, 2)	240
(5, 5, 2)	400
(6, 3, 3)	360
(5, 4, 3)	960
(6, 2, 4)	240
(5, 3, 4)	1200
(4, 4, 4)	1824
(5, 2, 5)	800
(4, 3, 5)	2160
(4, 2, 6)	1440
(3, 3, 6)	2520
(3, 2, 7)	1680
(2, 2, 8)	1120
(m, l, n)	$(\pi\omega\psi)$

40. The values of $(\pi\omega\psi)$ here tabulated will now enable us to apply the three formulæ of Art. 31 to the calculation of the number of special correlations of the second order which satisfy thirteen given elementary conditions.

Central-Axial Correlations.

41. To this end I consider, in the first place, the onefold system of central-axial correlations which satisfy the twelve elementary conditions whose signatures appear in the fourth column of the following table. The characteristics, $(\pi\psi)\,\mu$ and $(\pi\psi)\,\nu$, of this system are connected with its singularities, $(\pi\omega\psi)$, by means of the first relation of Art. 31, viz.:

$$2\,(\pi\psi)\,\mu = (\pi\psi)\,\nu + (\pi\omega\psi).$$

42. To determine the centre of a central-axial correlation, it is manifest that at least one plane, in which it is to lie, must appear amongst the data of each space, as also that not more than three such planes can do so. The values of the *second* element in the following signatures, therefore, must range between 2 and 6, and these signatures, accordingly, must constitute five distinct groups.

43. In each of these groups the initial value of the *first* element of the signature must be 2; for, although *three* pairs of points are necessary to determine the requisite projectivity of the pencils of planes about the two associated axes (Art. 19), one such pair, it must be remembered, is already indicated by the symbol at the head of the first column.

44. No such indication, however, attaches to the symbol $(\pi\psi)\,\nu$ at the head of the second column; so that an insufficiency occurs in the data of each of the initial cases of the above-mentioned five groups. This insufficiency, moreover, being, as usual, accompanied by an incompatibility, so far as the remaining conditions are concerned, no correlation is possible. Hence the zeros in the second column.

45. Finally, since of any two conjugate lines one must be incident with the axis of the space in which it lies, and since four lines at most can be so incident, it is manifest that the third element of no signature can exceed 8.

46. The circumstance that each number in the second column of the table is the same as that in the first column which is situated on the preceding line proceeds, of course, from the identity of the *thirteen* conditions which are fulfilled in the two cases.

47. Finally, it may be observed that here, as elsewhere, all cases are omitted which yield zero-values, exclusively, of the several terms *of the formula* under consideration.

$(\pi\psi)\,\mu$	$(\pi\psi)\,\nu$	$(\pi\omega\psi)$	Sig.
560	0	1120	2, 2, 8
1120	560	1680	3, 2, 7
1280	1120	1440	4, 2, 6
1040	1280	800	5, 2, 5
640	1040	240	6, 2, 4
320	640	0	7, 2, 3
160	320	0	8, 2, 2
80	160	0	9, 2, 1
40*	80	0	10, 2, 0
840	0	1680	2, 3, 7
1680	840	2520	3, 3, 6
1920	1680	2160	4, 3, 5
1560	1920	1200	5, 3, 4
960	1560	360	6, 3, 3
480	960	0	7, 3, 2
240	480	0	8, 3, 1
120	240	0	9, 3, 0
720	0	1440	2, 4, 6
1440	720	2160	3, 4, 5
1632	1440	1824	4, 4, 4
1296	1632	960	5, 4, 3
768	1296	240	6, 4, 2
384	768	0	7, 4, 1
192	384	0	8, 4, 0
400	0	800	2, 5, 5
800	400	1200	3, 5, 4
880	800	960	4, 5, 3
640	880	400	5, 5, 2
320	640	0	6, 5, 1
160	320	0	7, 5, 0
120	0	240	2, 6, 4
240	120	360	3, 6, 3
240	240	240	4, 6, 2
120	240	0	5, 6, 1
60†	120	0	6, 6, 0

* Seeing that the centre of this central-axial correlation may, in each space, lie in either of two given planes, we conclude, hence, that *there are* 20 *pairs of associated axes from which* 11 *pairs of points may be projected by* projective pencils of planes.

Planar-Axial Correlations.

48. Numerically, these do not differ from the correlations tabulated in the preceding Art. 47. By the Law of Duality, in fact, we may pass from the latter to the former without invoking the aid of the second formula in Art. 31.

Central-Planar Correlations.

49. The central-planar correlations now to be tabulated, like the central-axial ones of Art. 41, form a onefold system satisfying twelve elementary conditions. The characteristics $(\pi\omega)\,\nu$, $(\pi\omega)\,\mu$, $(\pi\omega)\,\rho$, and the singularities $(\pi\omega\psi)$, of the system are connected by the third relation of Art. 31, viz.:

$$2\,(\pi\omega)\,\nu = (\pi\omega)\,\mu + (\pi\omega)\,\rho + (\pi\omega\psi).$$

These characteristerics and singularities, in all cases except those yielding zero-results exclusively, are given in the first four columns of the following table.

The signatures of the data in the several cases are exhibited in the fifth column, where they form eleven groups, arranged according to ascending values of the third elements in such signatures.

For obvious reasons, neither the first nor the second element of any signature can exceed 6. The third, on the other hand, cannot be less than 2. For, in the first place, the singular planes have to contain projective pencils, whose corresponding rays have to be incident with conjugate lines; and, in the second place, the projectivity of these pencils can only be established by means of three pairs of corresponding elements—that is to say, two, in addition to the pair whose existence is indicated by the symbol $(\pi\omega)\,\nu$ at the head of the first column.

The occurrence of the zeros which appear in the second and third columns, and first five groups, is easily accounted for directly. The reappearance in the second and third columns of each group, after the first, of the numbers in the first column of the immediately preceding group, is due, of course, to the identity of the *thirteen* conditions fulfilled in the two cases.

This result agrees with one recorded by Sturm in his Memoir of 1873 on the "Problem of Projectivity in Space" (*Math. Ann.*, Bd. VI., p. 548).

† Although it would have been easy to verify, in a similar manner, all the results given by Sturm in the Memoir last referred to, one more will here suffice. Seeing that the centre of our central-axial correlation may here have, in each space, any one of $\frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3} = 20$ different positions, it follows that: *through fixed centres*, one in each space, *three* $(= \frac{60}{20})$ *different pairs of axes pass, whence seven given pairs of conjugate points can be projected by planes which form two projective pencils.* (See *Math. Ann., Bd. I, p. 544*, 1869; and Bd. VI., p. 530, 1873.)

With a view of economizing space, the expedient is again resorted to of providing the several columns with double headings, as in Art. 39.

The numbers of the fourth column are, of course, extracted from the Art. 89 last referred to.

$(\pi\omega)\,\nu$	$(\pi\omega)\,\mu$	$(\pi\omega)\,\rho$	$(\pi\omega\psi)$	Sig. $l,\ m,\ n$
120	0	0	240	4, 6, 2
200	0	0	400	5, 5, 2
240	120	0	360	3, 6, 3
640	200	120	960	4, 5, 3
240	240	0	240	2, 6, 4
1040	640	240	1200	3, 5, 4
1552	640	640	1824	4, 4, 4
120	240	0	0	1, 6, 5
1040	1040	240	800	2, 5, 5
2376	1552	1040	2160	3, 4, 5
60	120	0	0	0, 6, 6
580	1040	120	0	1, 5, 6
2428	2376	1040	1440	2, 4, 6
3636	2376	2376	2520	3, 3, 6
320	580	60	0	0, 5, 7
1504	2428	580	0	1, 4, 7
3872	3636	2428	1680	2, 3, 7
912	1504	320	0	0, 4, 8
2688	3872	1504	0	1, 3, 8
4432	3872	3872	1120	2, 2, 8
1800	2688	912	0	0, 3, 9
3560	4432	2688	0	1, 2, 9
2680	3560	1800	0	0, 2,10
3560	3560	3560	0	1, 1,10
3120	3560	2680	0	0, 1,11
3120	3120	3120	0	0, 0,12
$(\pi\omega)\,\nu$	$(\pi\omega)\,\rho$	$(\pi\omega)\,\mu$	$(\pi\omega\psi)$	$m,\ l,\ n$

50. From the last result the remarkable consequence is deduced which was first published by Sturm in his Memoir of 1877,* and which may be enunciated, succinctly, thus :—

Thirteen pairs of right lines being given, there exist 3120 *pairs of projective plane pencils, whereof, in each case, those rays correspond which are incident with the lines of any given pair.*

Half this number, viz. 1560, of pairs of projective plane pencils exist, when twelve pairs of lines are given, as well as a point through which the plane of *one* of the pencils of each pair must pass, or a plane in which the centre of *one* of them must lie (compare Art. 33).

Central Correlations.

51. In a onefold system of central correlations, satisfying thirteen conditions, we have to consider the three characteristics $\pi\mu$, $\pi\rho$, and $\pi\nu$, as well as the two singularities $(\pi\omega)$ and $(\pi\psi)$; between these five quantities two relations have been established (first and fifth, Art. 30) which are manifestly equivalent to the following :—

$$\left.\begin{aligned} 3\pi\mu &= \pi\rho + (\pi\psi) + 2(\pi\omega) \\ 3\pi\nu &= 2\pi\rho + 2(\pi\psi) + (\pi\omega) \end{aligned}\right\}.$$

52. The values of $(\pi\psi)$ and $(\pi\omega)$, in the several cases now to be tabulated, are extracted from the tables of Arts. 47 and 49; they vanish whenever the second element of the signatures, recorded in the last column of the subjoined table, exceeds 6; as do also, for equally obvious reasons, the values of $\pi\rho$, and with them, in virtue of the above two formulæ, the values of $\pi\mu$ and $\pi\nu$.

Arranged according to decreasing values, from 6 to 0, of the second elements of the signatures, therefore, the following table consists of seven parts or groups; wherein, it will be observed, the values in the third column of any group have, necessarily, already appeared in the second and first columns of the preceding group, since the *fourteen* conditions, satisfied in both cases, are the same.

$\pi\mu$	$\pi\nu$	$\pi\rho$	$(\pi\psi)$	$(\pi\omega)$	Sig.
40	20	0	0	60	0, 6, 7
80	40	0	0	120	1, 6, 6
160	80	0	0	240	2, 6, 5
200	160	0	120	240	3, 6, 4
160	200	0	240	120	4, 6, 3
80	160	0	240	0	5, 6, 2
40	80	0	120	0	6, 6, 1
20	40	0	60	0	7, 6, 0

* "*Ueber correlative oder* reciproke Bündel," *Math. Ann.*, Vol. XII., p. 363.

$\pi\mu$	$\pi\nu$	$\pi\rho$	$(\pi\psi)$	$(\pi\omega)$	Sig.
220	120	20	0	320	0, 5, 8
400	220	40	0	580	1, 5, 7
720	400	80	0	1040	2, 5, 6
880	720	160	400	1040	3, 5, 5
760	880	200	800	640	4, 5, 4
480	760	160	880	200	5, 5, 3
240	480	80	640	0	6, 5, 2
120	240	40	320	0	7, 5, 1
60	120	20	160	0	8, 5, 0
648	384	120	0	912	0, 4, 9
1076	648	220	0	1504	1, 4, 8
1752	1076	400	0	2428	2, 4, 7
2064	1752	720	720	2376	3, 4, 6
1808	2064	880	1440	1552	4, 4, 5
1224	1808	760	1632	640	5, 4, 4
672	1224	480	1296	120	6, 4, 3
336	672	240	768	0	7, 4, 2
168	336	120	384	0	8, 4, 1
84	168	60	192	0	9, 4, 0
1328	856	384	0	1800	0, 3, 10
2008	1328	648	0	2688	1, 3, 9
2940	2008	1076	0	3872	2, 3, 8
3288	2940	1752	840	3636	3, 3, 7
2832	3288	2064	1680	2376	4, 3, 6
1936	2832	1808	1920	1040	5, 3, 5
1088	1936	1224	1560	240	6, 3, 4
544	1088	672	960	0	7, 3, 3
272	544	336	480	0	8, 3, 2
136	272	168	240	0	9, 3, 1
68	136	84	120	0	10, 3, 0
2072	1464	856	0	2680	0, 2, 11
2816	2072	1328	0	3560	1, 2, 10
3624	2816	2008	0	4432	2, 2, 9
3748	3624	2940	560	3872	3, 2, 8
3088	3748	3288	1120	2428	4, 2, 7
2064	3088	2832	1280	1040	5, 2, 6
1152	2064	1936	1040	240	6, 2, 5
576	1152	1088	640	0	7, 2, 4
288	576	544	320	0	8, 2, 3
144	288	272	160	0	9, 2, 2
72	144	136	80	0	10, 2, 1
36	72	68	40	0	11, 2, 0

$\pi\mu$	$\pi\nu$	πo	$(\pi\psi)$	$(\pi\omega)$	Sig.
2568	2016	1464	0	3120	0, 1, 12
3064	2568	2072	0	3560	1, 1, 11
3312	3064	2816	0	3560	2, 1, 10
3000	3312	3624	0	2688	3, 1, 9
2252	3000	3748	0	1504	4, 1, 8
1416	2252	3088	0	580	5, 1, 7
768	1416	2064	0	120	6, 1, 6
384	768	1152	0	0	7, 1, 5
192	384	576	0	0	8, 1, 4
96	192	288	0	0	9, 1, 3
48	96	144	0	0	10, 1, 2
24	48	72	0	0	11, 1, 1
12	24	36	0	0	12, 1, 0
2752	2384	2016	0	3120	0, 0, 13
2936	2752	2568	0	3120	1, 0, 12
2808	2936	3064	0	2680	2, 0, 11
2304	2808	3312	0	1800	3, 0, 10
1608	2304	3000	0	912	4, 0, 9
964	1608	2252	0	320	5, 0, 8
512	964	1416	0	60	6, 0, 7
256	512	768	0	0	7, 0, 6
128	256	384	0	0	8, 0, 5
64	128	192	0	0	9, 0, 4
32	64	96	0	0	10, 0, 3
16	32	48	0	0	11, 0, 2
8	16	24	0	0	12, 0, 1
*4	8	12	0	0	13, 0, 0

* The numbers recorded in the first column of the above Table, though opposite in their order of sequence, agree perfectly with those given by Sturm in the Memoir, of 1877, last cited. To render this agreement manifest, however, certain factors have to be applied to Sturm's results, as shall now be briefly indicated for each of the seven groups which the table includes.

Group 1.—The numbers here are equi-multiples of those quoted by Sturm, in his Art. 40 (p. 291), from my paper of 1874 "On the the Correlation of Two Planes" (*Proc. of the Math. Soc.*, Vol. v., Art. 41). The factor applied is $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$; this being the number of different positions which the associated centres of our central correlation may assume, when each centre lies in three of the six given planes, situated in each of the correlated spaces.

Group 2.—These numbers are equi-multiples of those recorded in Sturm's Table 1, Art. 40 (p. 291). The factor 20 is again applied to the latter; but this factor is now due to the circumstance that one centre may lie either in three, or in two, of five given planes; its associate in the correlated space being situated in the planes conjugate to the remaining two, or three.

Planar Correlations.

53. By the law of duality, planar correlations are immediately deducible from the central ones last tabulated. They do not, consequently, require separate treatment.

Axial Correlations.

54. In a onefold system of axial correlations satisfying thirteen conditions, we have again to consider three characteristics and two singularities. The former are $\psi\mu$, $\psi\rho$, and $\psi\nu$; and the latter $(\omega\psi)$ and $(\pi\psi)$. These five quantities are connected by the second and fourth relations of Art. 30; since the two latter, however, are themselves related by the law of duality, it will suffice to apply the one last mentioned, viz.:

$$2\psi\rho = \psi\nu + (\pi\psi).$$

55. In the following Table the signatures of the thirteen given conditions are arranged according to descending values of the third element thereof, that is to say, of the one which gives the number of given pairs of conjugate lines. Moreover, since of any two lines forming such a pair one must necessarily be incident with an axis of the

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Group 3.—These numbers are obtained by adding to 8 times each of those given in Sturm's Table 2 of Art. 55 (p. 309), 6 times each of the numbers in his Table 3, of Art. 64 (p. 320). The factor 8 is accounted for by the fact that in *each of the two* spaces a centre may lie in any one of four given planes, while its associate, in the correlated space, is fixed by being, in each of the planes, conjugate to the remaining three. The factor 6, on the other hand, is due to the circumstance that the centre, in either of the two spaces, may lie in any two of four planes, while its associate is situated in both the planes conjugate to the remaining two.

Group 4.—Twice the several numbers given in Sturm's Table 4 of Art. 69 (p. 325), added to six times those recorded in his Table 5 of Art. 80 (p. 335), give the numbers now in question. The factor 2 presents itself because one of the associated centres may be fixed by being situated on all the three given planes of either of the two spaces; the factor 6, on the other hand, because a centre may lie in any one of the three planes given in *each* of the two spaces, while its associate is situated in both the planes conjugate to the remaining two.

Group 5.—The numbers here are obtained by adding to twice those contained in Sturm's Table 6, of Art. 90 (p. 345), twice those given in his Table 7 of Art. 97 (p. 352). The first factor 2 is due to the circumstance that in either of the two spaces the centre may be situated on both the planes which that space contains; the second factor 2 is accounted for by the fact that in each space the centre may lie in either of two given planes while its associate is situated in the plane conjugate to the other.

Group 6.—These numbers are the doubles of those given by Sturm in his Table 8 of Art. 103 (p. 358); since the centre, either in one or the other of the two spaces, must lie in a given plane; its associate being determined, of course, by the remaining twelve conditions.

Group 7.—These are precisely the numbers given by Sturm in the Table 9 of his Art. 108 (p. 365); the centres being no longer subject to the condition of lying in any given planes.

correlation, the greatest possible value of the element in question is $8 = 2 \times 4$.

On the other hand, the least value of the first element of the signature is 3, seeing that this number of given pairs of conjugate points is necessary to the establishment of the projectivity of the pencils of planes about the two associated axes.

The fact, moreover, that three given pairs of conjugate planes are necessary for the determination of the projectivity of the two ranges of points on the axes in question, accounts for the circumstance that the value of $\psi\nu$ is 0, which terminates each of the nine groups of the following Table.

After the explanations given in Arts. 49 and 52, the repetition, in the second column, of numbers which have already appeared in the first, requires no further remark.

The values of $(\pi\psi)$, it will be observed, are extracted from the Table of Art. 47.

$\psi\rho$	$\psi\nu$	$(\pi\psi)$	Sig.
280	0	560	3, 2, 8
560	280	840	3, 3, 7
560	0	1120	4, 2, 7
640	560	720	3, 4, 6
1120	560	1680	4, 3, 6
640	0	1280	5, 2, 6
520	640	400	3, 5, 5
1280	1120	1440	4, 4, 5
1280	640	1920	5, 3, 5
520	0	1040	6, 2, 5
320	520	120	3, 6, 4
1040	1280	800	4, 5, 4
1456	1280	1632	5, 4, 4
1040	520	1560	6, 3, 4
320	0	640	7, 2, 4
160	320	0	3, 7, 3
640	1040	240	4, 6, 3
1168	1456	880	5, 5, 3
1168	1040	1296	6, 4, 3
640	320	960	7, 3, 3
160	0	320	8, 2, 3

$\psi\rho$	$\psi\nu$	$(\pi\psi)$	Sig.
80	160	0	3, 8, 2
320	640	0	4, 7, 2
704	1168	240	5, 6, 2
904	1168	640	6, 5, 2
704	640	768	7, 4, 2
320	160	480	8, 3, 2
80	0	160	9, 2, 2
40	80	0	3, 9, 1
160	320	0	4, 8, 1
352	704	0	5, 7, 1
512	904	120	6, 6, 1
512	704	320	7, 5, 1
352	320	384	8, 4, 1
160	80	240	9, 3, 1
40	0	80	10, 2, 1
20	40	0	3, 10, 0
80	160	0	4, 9, 0
176	352	0	5, 8, 0
256	512	0	6, 7, 0
286	512	60	7, 6, 0
256	352	160	8, 5, 0
176	160	192	9, 4, 0
80	40	120	10, 3, 0
20	0	40	11, 2, 0*

Ordinary Correlations.

56. We are at length in a position to treat the general question concerning ordinary correlations which satisfy fifteen elementary conditions,—the question referred to at the commencement of the present paper.

This is done by considering, in the first place, the onefold system of correlations that satisfy the several sets of fourteen conditions whose signatures are given in the seventh column of the following Table.

* This verifies a theorem given by Sturm in his paper of 1873, "On Projectivity in Space" (*Math. Ann.*, Vol. VI., p. 548). According to it, in fact, there exist 20 pairs of associated axes, whence 11 pairs of given points may be projected by means of two projective pencils of planes.

Several other theorems, indeed, given by Sturm in the paper in question, are confirmed by the preceding results.

These signatures, it will be observed, are arranged according to ascending values, 0 to 14, of their third elements, and they thus form fifteen groups or sections.

The characteristics μ, ρ, ν of the system, and their singularities π, ω, ψ, have been already defined (see Arts. 2 and 3). The three relations, moreover, between these six quantities have been already given in Art. 24; the characteristics, indeed, have been actually expressed in terms of the singularities in Art. 1.

The values of π in the fourth column will be found, under their appropriate signatures, in the Table of Art. 52. The values of ω, in the fifth column, follow from the latter by the law of duality. The values of ψ, finally, recorded in the sixth column, are extracted from the Table of Art. 55.

57. For economy of space, the method of double entry (see Arts. 39 and 49) has, in the following Table, been once more employed.

μ	ρ	ν	π	ω	ψ	Sig. l, m, n
*1	3	2	4	0	0	14, 0, 0
3	9	6	12	0	0	13, 1, 0
9	27	18	36	0	0	12, 2, 0
27	61	54	68	0	20	11, 3, 0
61	103	122	84	0	80	10, 4, 0
103	133	206	60	0	176	9, 5, 0
133	143	266	20	0	256	8, 6, 0
143	143	286	0	0	286	7, 7, 0
2	6	4	8	0	0	13, 0, 1
6	18	12	24	0	0	12, 1, 1
18	54	36	72	0	0	11, 2, 1
54	122	108	136	0	40	10, 3, 1
122	206	244	168	0	160	9, 4, 1
206	266	412	120	0	352	8, 5, 1
266	286	532	40	0	512	7, 6, 1
ρ	μ	ν	ω	π	ψ	m, l, n

* From this result we infer that, whenever fourteen pairs of conjugate points are given, there is *one* correlation in which a given fifteenth pair of points are likewise *conjugate, three* correlations in which two given planes are conjugate to one another, *and two correlations in* which a given pair of lines are so.

μ	ρ	ν	ϖ	ω	ψ	Sig. l, m, n
4	12	8	16	0	0	12, 0, 2
12	36	24	48	0	0	11, 1, 2
36	108	72	144	0	0	10, 2, 2
108	244	216	272	0	80	9, 3, 2
244	412	488	336	0	320	8, 4, 2
412	532	824	240	0	704	7, 5, 2
532	532	984	80	80	904	6, 6, 2
8	24	16	32	0	0	11, 0, 3
24	72	48	96	0	0	10, 1, 3
72	216	144	288	0	0	9, 2, 3
216	488	432	544	0	160	8, 3, 3
488	824	976	672	0	640	7, 4, 3
824	984	1488	480	160	1168	6, 5, 3
16	48	32	64	0	0	10, 0, 4
48	144	96	192	0	0	9, 1, 4
144	432	288	576	0	0	8, 2, 4
432	976	864	1088	0	320	7, 3, 4
976	1488	1752	1224	200	1040	6, 4, 4
1488	1488	2216	760	760	1456	5, 5, 4
32	96	64	128	0	0	9, 0, 5
96	288	192	384	0	0	8, 1, 5
288	864	576	1152	0	0	7, 2, 5
864	1752	1568	1936	160	520	6, 3, 5
1752	2216	2624	1808	880	1280	5, 4, 5
64	192	128	256	0	0	8, 0, 6
192	576	384	768	0	0	7, 1, 6
576	1568	1072	2064	80	0	6, 2, 6
1568	2624	2416	2832	720	640	5, 3, 6
2624	2624	3184	2064	2064	1120	4, 4, 6
128	384	256	512	0	0	7, 0, 7
384	1072	728	1416	40	0	6, 1, 7
1072	2416	1744	3088	400	0	5, 2, 7
2416	3184	3080	3288	1752	560	4, 3, 7

ρ	μ	ν	ω	π	ψ	m, l, n

μ	ρ	ν	π	ω	ψ	Sig. l, m, n
256	728	492	964	20	0	6, 0, 8
728	1744	1236	2252	220	0	5, 1, 8
1744	3080	2412	3748	1076	0	4, 2, 8
3080	3080	3220	2940	2940	280	3, 3, 8
492	1236	864	1608	120	0	5, 0, 9
1236	2412	1824	3000	648	0	4, 1, 9
2412	3220	2816	3624	2008	0	3, 2, 9
864	1824	1344	2304	384	0	4, 0, 10
1824	2816	2320	3312	1328	0	3, 1, 10
2816	2816	2816	2816	2816	0	2, 2, 10
1344	2320	1832	2808	856	0	3, 0, 11
2320	2816	2568	3064	2072	0	2, 1, 11
1832	2568	2200	2936	1464	0	2, 0, 12
2568	2568	2568	2568	2568	0	1, 1, 12
2200	2568	2384	2752	2016	0	1, 0, 13
2384	2384	2384	2384	2384	0	0, 0, 14
ρ	μ	ν	ω	π	ψ	m, l, n

58. The greatest number of correlations, satisfying fifteen conditions, recorded in the above Table, is 3220. It presents itself when the data are: three pairs of conjugate points, three pairs of conjugate planes, and nine pairs of conjugate lines.

Another notable peculiarity is the repetition, in the first five columns, of the number 2816, when the signature is (2, 2, 10), and of 2384 when the signature reaches its final limit (0, 0, 14).

Multiple Conditions.

59. I do not propose to give any direct verification of the above results, nor will I extend this paper to greater length by any attempt to treat the very numerous cases of correlation under mixed, *multiple as well as single*, conditions.

It is true that, when *three*, at least, of the latter description are involved, the above method requires but little modification. But, when less than three single conditions are given, a mode of transformation of multiple into equivalent single conditions may become necessary;—unless, indeed, the problem admits of solution by a more direct and elementary method.

60. By way of conclusion, in fact, a few words may be here added, relative to cases of the latter kind,—cases wherein *all* the conditions are triple ones, and where the corresponding data consist, exclusively, of arbitrarily situated points and planes.

61. There are, obviously, only the following three essentially different cases of the above character, viz.:

First. Five given points in one space, corresponding to five given planes in the other.

Secondly. Four points and one plane, corresponding, respectively, to four planes and one point.

Thirdly. Three points and two planes, in one of the two spaces, corresponding, respectively, to three planes and two points in the other.

In each of the first two of these three cases, there is one and only one solution. In the third there is, in general, no solution whatever.

62. Placing, in fact, each of the given points A, B, C, D, E in vertical line with its corresponding plane $\alpha, \beta, \gamma, \delta, \epsilon$, we have to consider the following three arrangements:—

$$\left|\begin{matrix} A, & B, & C, & D, & E \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\left|\begin{matrix} A, & B, & C, & D, & \epsilon \\ \alpha & \beta, & \gamma, & \delta, & E \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\left|\begin{matrix} A, & B, & C, & \delta, & \epsilon \\ \alpha, & \beta, & \gamma, & D, & E \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

63. In the *first* of these cases, each of the sides of the trilateral ABC in (1) may be taken as the axis of a pencil of planes, projective with the row of points on the corresponding edge of the trihedral $\alpha\beta\gamma$. To determine the projectivity, we have the plane (ABC), common to all three pencils, corresponding to the point $(\alpha\beta\gamma)$, common to all three rows; moreover, the planes in each pencil which pass through D and E correspond, respectively, to the intersections of δ and ϵ with the axis of the associated row.

This projectivity being fixed, to any point P, regarded as the intersection of three planes, of which one belongs to each pencil, will correspond the determinate plane π which passes through the three points corresponding to those planes.

The correlation, accordingly, is fixed and unique.

64. With respect to the *second* of the above three arrangements of data, it is to be observed that, from what has just been proved, we may infer that there is one, and only one, correlation in which the five planes

$$(BCD),\quad (CDA),\quad (DAB),\quad (ABC),\quad \epsilon$$

correspond, respectively, to the five points

$$(\beta\gamma\delta),\quad (\gamma\delta\alpha),\quad (\delta\alpha\beta),\quad (\alpha\beta\gamma),\quad E.$$

But this correlation satisfies the condition (2) of Art. 62, with which we are at present concerned; seeing, *e.g.*, that the point A, common to the above second, third, and fourth planes, necessarily corresponds to the plane α which passes through the second, third, and fourth of the points just indicated. Conversely, any correlation which satisfies the conditions (2), of Art. 62, clearly satisfies the five last referred to.

The solution arrived at is, accordingly, unique.

65. I observe, *lastly*, that if it were possible to satisfy the group (3) of five conditions, indicated in Art. 62, by any correlation, we should necessarily have a projectivity established between the pencil of planes

$$\overline{\delta\epsilon}\,(A, B, C, \delta, \epsilon),$$

and the row of points

$$\overline{DE}\,(\alpha, \beta, \gamma, D, E).$$

But this projectivity being, manifestly, inconsistent with the predicated arbitrariness, in position, of the given points and planes, we conclude that, in general, no correlation is possible which satisfies the five conditions (3) of Art. 62.

On the Deformation of an Elastic Shell.

By Professor HORACE LAMB, M.A., F.R.S.

[*Read Jan. 9th*, 1890.]

The following paper treats of the deformation of an elastic shell whose radii of curvature are everywhere great in comparison with the thickness, which is supposed uniform. This subject has been investigated in a very able manner by Mr. A. E. H. Love, in a recent paper,* but it appeared desirable on various grounds that the problem should be attacked from an independent point of view. The ideas which form the basis of the method here followed have been explained with some fulness in a previous communication, "On the Flexure of an Elastic Plate";† I have therefore contented myself on the present occasion with very slight indications in this respect. I may, however, recall the fundamental principle that, if the external forces on each element of the shell are by themselves in equilibrium, the deformation is insensible, as also the resultant stress across an element of any normal section. By an *element of the shell* is here meant the portion cut off by a ruled surface whose generators are normals to the middle surface, drawn through the contour of a small area of dimensions comparable with the thickness. In the same way, by an *element of a normal section* is meant the section generated by the normals to the middle surface at all points of a linear element of like order of magnitude.

As regards the general theory, my results show a close agreement with those arrived at by Mr. Love, and the investigation may be of interest as giving a physical meaning to the various groups of terms which enter into his equations. There are, however, some points of difference; these appear to be due to a somewhat different definition of the symbols which express the deformation of the middle surface, and are practically unimportant.

One reason of the great complexity of our present subject, as contrasted with the theory for a plane plate, is that it is not possible to draw an absolute line of demarcation between the deformations in which the governing feature is the extension of the middle surface, and those which involve flexure with little or no extension. To illustrate this we may take the case of a spherical bowl, of thickness small compared with the radius, whose edge is a small circle. If the

* *Phil. Trans.*, 1888 (A), p. 491. † *Proc. Lond. Math. Soc.*, Dec. 12, 1889, p. 70.

angular radius of the bowl is very small, its behaviour under various applications of force, and its vibrations when free, cannot differ materially from those of a plane circular disk. In particular, there will be modes of (mainly) flexural vibration whose frequency is proportional to the thickness. By increasing the angular radius, the thickness being constant, we obtain a continuous succession of forms leading up to the extreme case of a spherical shell with a vanishing circular aperture. In this extreme, it is known, no deformation is possible without extension of the middle surface, and, if the shell be sufficiently thin, this extension is the governing element in its behaviour. Thus the frequencies of all the modes are now independent of the thickness, and, accordingly, great in comparison with those of the flexural modes of a plane plate of similar dimensions. Between these two extremes we have, of course, all intermediate stages.*

The main source of this complexity appears to lie in the fact, pointed out by Mr. Love, that it is in general impossible to satisfy the boundary conditions by a deformation such that the middle surface is

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[* *February*, 1890. It is evident (as pointed out by one of the referees) that we may at any stage restore the flexural character of the vibrations, however small the aperture, by sufficiently reducing the thickness; but in the above argument the aperture only is supposed to vary. To examine more closely the transition to an infinitely small aperture, we may make use of the formulæ given by Lord Rayleigh in his investigation of the deformation of a spherical bowl, on the hypothesis of no extension (*Proceedings*, T. XIII., p. 4). It appears that, when the angular radius (α) of the aperture is small, the kinetic energy in the mode expressed by circular functions of $s\omega$ (where ω is the longitude, and s is integral and > 1) is

$$T = \frac{\pi}{2(s-1)} \rho a^2 h \alpha^2 . \dot{B}_s^2,$$

where B_s is the maximum radial amplitude, and the rest of the notation is as in this paper. For the potential energy we find

$$V = \tfrac{1}{4}\pi s^2 (s+1) \mu \frac{h^3}{a^2 \alpha^2} . B^2.$$

Hence, if the time-factor be $e^{ip_s t}$, we have

$$p_s^2 = \tfrac{1}{4} s^2 (s^2-1) \frac{\mu}{\rho} \cdot \frac{h^2}{a^4 \alpha^4},$$

which increases continually as α diminishes. In the gravest corresponding mode for the complete spherical shell we have

$$p_s'^2 = C\mu/\rho a^2,$$

where C is a numerical factor (*Proceedings*, T. XIV., p. 50). The ratio p_s/p_s' is of the order $ha/a^2\alpha^2$, *i.e.*, it varies as the square of the ratio which the mean proportional $\sqrt{ah}$ bears to the *linear* radius of the aperture; so that, if α is sufficiently diminished, p_s becomes at length greater than p_s'. With a sufficiently small value of h/a this may happen while the breadth of the aperture is still a considerable multiple of the thickness. It is evident that, before this stage is reached, the hypothesis of no extension will have ceased to give an adequate representation of the actual vibration-

absolutely unextended.* But, the existence of extension as a general accompaniment of the flexure of a curved shell being admitted, the question remains, in any particular problem, as to its order of magnitude and its distribution, and more especially whether there are any modes of deformation (or of free vibration) in which, after all, it plays only a secondary part. Mr. Love, starting from the expression for the potential energy per unit area in the form†

$$W = 3Ah(\sigma_1^2 + 2\sigma\sigma_1\sigma_2 + \sigma_2^2) + \mu h\varpi^2 + Ah^3(\Sigma_1^2 + 2\sigma\Sigma_1\Sigma_2 + \Sigma_2^2) + \tfrac{1}{3}\mu h^3\Pi^2,$$

where A and μ are certain elastic constants, h is the half-thickness, whilst σ_1, σ_2, ϖ define the extension of the middle surface, and Σ_1, Σ_2, Π its change of curvature, argues that, since (as we have seen) the quantities σ_1, σ_2, ϖ are not zero, the second line will ultimately (when h is sufficiently diminished) vanish in comparison with the first, and that, accordingly, the extension is the cardinal feature in all questions relating to the deformation of very thin curved shells. Without stopping to develop the consequences of this position, which appear to be in conflict not merely with popular notions on the subject, but with established dynamical principles,‡ it may be observed that, if the extensional strains (σ_1, σ_2, ϖ) are comparable with those ($h\Sigma_1$, $h\Sigma_2$, $h\Pi$) due to flexure, the second (or flexural) part of W will be of the same order as the first; moreover, it is conceivable that the extensions may be practically confined to so small a region that, when integrated over the whole area of the shell, the flexural part greatly preponderates.

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type. In the view taken by the writer, extensions are called into play (at first in the neighbourhood of the edge only, where the flexure is greatest) which lead to a greater kinetic and a smaller potential energy, for assigned values of B_1 and B_2 respectively, than are given by the preceding formulæ, and in this way the vibration-types and the frequencies become continuous with those proper to the complete sphere. It is known that in this latter case there are two distinct modes of vibration corresponding to any given spherical harmonic, and that one of these has a much lower frequency than the other. These appear to be the homologues of flexural and extensional modes, respectively, of a plane circular disk.]

* There is another and more obvious reason, independent of the boundary conditions, why extensions should, as a rule, be associated with the flexure of a curved shell (or bar). In statical questions the curvature itself may cause a normal force applied at one point to be transmitted (in part at least) as a tangential stress to another point, as, for example, in the case of a cylindrical shell (or a circular ring) bent by two equal and opposite forces applied at the ends of a diameter. In the case of free vibrations, a similar effect is produced by inertia: this is illustrated in § 10, below. In most cases of interest, however, the extensional strains hence arising appear to be small in comparison with those referred to in the text.

† The notation is that of this paper, and the meanings of Σ_1, Σ_2, Π are slightly different from those of the corresponding symbols in Mr. Love's formula; the argument is, however, not affected.

‡ See Lord Rayleigh, *Proc. Roy. Soc.*, Dec. 13, 1888.

In order to bring the matter to a test, in a definite instance, I have taken the case of a cylindrical plate (such as a boiler plate) bent by a proper application of force over its straight edges so that the strained form is one of revolution, the circular edges being free. The analytical work is very simple, and the physical meaning of the various terms which occur is easily recognised; so that this part of the paper may be followed without difficulty by readers who may not care to occupy themselves with the somewhat intricate investigations which precede. In the interpretation of the results, a good deal turns on the ratio which the breadth of the plate (in the direction of the generating lines) bears to a mean proportional between the radius and the thickness. When this ratio is large, the bending forces may be practically replaced by equal and opposite couples uniformly distributed over the two straight edges, and having these edges as axes. The strained form is almost accurately cylindrical; near the edge we have extensions of the same order as the strains due to change of curvature, but they rapidly diminish (at the same time fluctuating in sign) as we pass inwards, and their contribution to the total energy is unimportant. In such a case the approximate method employed by Lord Rayleigh, in which no account is taken of the conditions at a free edge, is fully justified. But if, keeping the radius and the thickness constant, we diminish the breadth until it is comparable with the mean proportional aforesaid, we get a sort of transitional stage between a plate and a bar which cannot be satisfactorily dealt with except by the general equations. Finally, when the breadth is small compared with the mean proportional, the plate behaves like a curved bar, and an approximate treatment is again applicable.

1. As in the former paper, we first consider a state of strain such that, any arbitrary deformation having been given to the middle surface, a line of particles originally normal to this surface remains straight and normal to it in the strained condition, whilst the elongation in the direction of the normal is adjusted so as to give zero traction across any small area parallel to the middle surface. The work is, however, now more complicated on account of the more general character of the deformations of the middle surface which we have to consider.

Let us take as temporary coordinate axes the tangents to the two lines of curvature, and the normal, at any point P of the middle surface in the unstrained condition. For simplicity of conception, we may regard these axes as fixed, and imagine that, after the deformation, the surface is moved as a whole until the point P is brought back to the origin, and the tangent plane into coincidence

with the plane xy in such a position that the *rotation* of an element of area on the middle surface at P is zero. Hence, if (ξ, η, ζ) and (ξ', η', ζ') be the coordinates, before and after the deformation, of any the same point Q of the middle surface near P, we shall have

$$\xi' = (1+\sigma_1)\,\xi + \tfrac{1}{2}\varpi\eta, \quad \eta' = \tfrac{1}{2}\varpi\xi + (1+\sigma_2)\,\eta,$$

where σ_1, σ_2 are the extensions parallel to x, y, and ϖ is the shear in the tangent plane. Again, the shape of the middle surface before and after strain will be given by equations of the forms

$$2\zeta = A\xi^2 + B\eta^2 + \&c.,$$
$$2\zeta' = A'\xi^2 + 2H'\xi\eta' + B'\eta^2 + \&c.$$

The deformation of the middle surface at P is completely specified by the six quantities σ_1, σ_2, ϖ, $A'-A$, $B'-B$, H'. It is evident that (to the first order) $A'-A$, $B'-B$ are the changes in the values of the principal curvatures, whilst H' determines the change of direction (relative to the substance) of the lines of curvature.

In some subsequent calculations we shall require the value of ζ' in terms of ξ, η. We find

$$2\zeta' = A_1\xi^2 + 2H_1\xi\eta + B_1\eta^2 + \&c.,$$

where $A_1 = A' + 2\sigma_1 A$, $\quad H_1 = H' + \tfrac{1}{2}\varpi\,(A+B)$, $\quad B_1 = B' + 2\sigma_2 B$,

squares and products of strains being neglected.

The coordinates of a point of the substance at a distance z along the normal at Q were originally

$$x = \xi - z\frac{d\zeta}{d\xi}, \quad y = \eta - z\frac{d\zeta}{d\eta}, \quad z.$$

In the state of strain above defined, these become

$$x' = \xi' - z'\frac{d\zeta'}{d\xi'}, \quad y' = \eta' - z'\frac{d\zeta'}{d\eta'}, \quad z' = \int_0^z (1+e_{zz})\,dz,$$

where e_{zz} is the elongation in the direction of the normal. Hence, for the component displacements, we find

$$\alpha = \sigma_1\xi + \tfrac{1}{2}\varpi\eta - z\left\{(A'-A)\,\xi + H'\eta + A\,(\sigma_1\xi + \tfrac{1}{2}\varpi\eta)\right\} - \int_0^z e_{zz}\,dz\,.\,A\xi,$$

$$\beta = \tfrac{1}{2}\varpi\xi + \sigma_2\eta - z\left\{H'\xi + (B'-B)\,\eta + B\,(\tfrac{1}{2}\varpi\xi + \sigma_2\eta)\right\} - \int_0^z e_{zz}\,dz\,.\,B\eta,$$

$$\gamma = \int_0^z e_{zz}\,dz.$$

These may be simplified by neglecting $zA \,.\, \sigma_1$ in comparison with σ_1, and so on. Since e_{zz} is of the same order as the remaining strains, the last term in α and β may also be omitted for the same reason. Again, in finding the strains parallel to the plane xy, we have to differentiate α, β with respect to x and y. Since, however,

$$\xi = x/(1-zA), \quad \eta = y/(1-zB),$$

we may, on the same grounds, differentiate with respect to ξ, η instead. In this way we find, at points on the axis of z,

$$e_{xx} = \frac{d\alpha}{d\xi} = \sigma_1 - \Sigma_1 z,$$

$$e_{yy} = \frac{d\beta}{d\eta} = \sigma_2 - \Sigma_2 z,$$

$$e_{xy} = \frac{d\alpha}{d\eta} + \frac{d\beta}{d\xi} = \varpi - \Pi z,^*$$

where $$\Sigma_1 = A' - A, \quad \Sigma_2 = B' - B, \quad \Pi = 2H'.$$

The approximations made above show that a certain latitude is admissible in the definition of the coefficients Σ_1, Σ_2, Π; for example, we might with equal propriety write A_1, B_1, H_1 in place of A', B', H'. This is to be borne in mind when comparing the values found for these coefficients in § 7, below, with the corresponding expressions in Mr. Love's memoir.

Finally, we have

$$e_{xz} = \frac{d\alpha}{dz} + \frac{d\gamma}{d\xi} = -\int_0^z \frac{de_{xx}}{d\xi} dz,$$

$$e_{yz} = \frac{d\beta}{dz} + \frac{d\gamma}{d\eta} = -\int_0^z \frac{de_{zz}}{d\eta} dz,$$

where $\xi = 0$, $\eta = 0$. These strains may be neglected in comparison with the former, if (as we shall suppose) the variations of σ_1, σ_2, &c., from point to point of the middle surface, within distances comparable with h, are only small fractions of these functions themselves.†

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* It may be noted that a small area parallel to the middle surface, at a distance z from it, undergoes a relative rotation above the normal, of amount

$$\tfrac{1}{2}\left(\frac{d\beta}{d\xi} - \frac{d\alpha}{d\eta}\right) = \tfrac{1}{2}\varpi\,(A-B)\,z.$$

† This, with a like assumption in § 3, is merely provisional, and may be tested *à posteriori* in any particular application of the theory.

2. With a usual notation for the stresses and elastic constants, we have

$$p_{xx} = (\lambda+2\mu)\, e_{xx}+\lambda\,(e_{yy}+e_{zz}),$$

$$p_{yy} = (\lambda+2\mu)\, e_{yy}+\lambda\,(e_{zz}+e_{xx}),$$

$$p_{zz} = (\lambda+2\mu)\, e_{zz}+\lambda\,(e_{xx}+e_{yy}).$$

By hypothesis, e_{zz} is chosen so as to make $p_{zz} = 0$; substituting in the two former equations, we obtain

$$p_{xx} = \frac{4\,(\lambda+\mu)\,\mu}{\lambda+2\mu}\,(e_{xx}+\sigma e_{yy}),$$

$$p_{yy} = \frac{4\,(\lambda+\mu)\,\mu}{\lambda+2\mu}\,(e_{yy}+\sigma e_{xx}),$$

where σ, $= \lambda/2\,(\lambda+\mu)$, is Poisson's ratio. To these we must join

$$p_{xy} = \mu e_{xy}.$$

We can hence calculate the potential energy W per unit area of the middle surface. The energy per unit volume is

$$\tfrac{1}{2}\,(p_{xx}e_{xx}+p_{xy}e_{xy}+p_{yy}e_{yy})$$

$$= \tfrac{1}{2}\left[\frac{4\,(\lambda+\mu)\,\mu}{\lambda+2\mu}\,\{e_{xx}^2+2\sigma e_{xx}e_{yy}+e_{yy}^2\}+\mu e_{xy}^2\right].$$

To find W we must integrate this with respect to z between the limits $\pm h$,* where h is the half-thickness. Substituting the values of e_{xx}, e_{xy}, e_{yy}, from the preceding paragraph, we obtain

$$2W = \left[\frac{4\,(\lambda+\mu)\,\mu}{\lambda+2\mu}\,\{\sigma_1^2+2\sigma.\sigma_1\sigma_2+\sigma_2^2\}+\mu\varpi^2\right]2h$$

$$+\left[\frac{4\,(\lambda+\mu)\,\mu}{\lambda+2\mu}\,\{\Sigma_1^2+2\sigma\Sigma_1\Sigma_2+\Sigma_2^2\}+\mu\Pi^2\right]\frac{2h^3}{3}.$$

Let us next find the resultant stresses across normal sections of the plate. Let P_{xx}, P_{xy} be the components parallel to x, y of the force-resultant of the stress per unit length (measured on the middle surface) of a normal section perpendicular to x, whilst G_{xx}, G_{xy} are

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* In strictness, we should multiply by $(1-zA)(1-zB)$ before integrating; but the error is unimportant.

the components of the resultant couple; and let a similar notation be used for a normal section perpendicular to y. We have

$$P_{xx} = \int_{-h}^{h} p_{xx}\,dz = \frac{8(\lambda+\mu)\mu}{\lambda+2\mu} h\,(\sigma_1+\sigma\sigma_2),$$

$$P_{yy} = \int_{-h}^{h} p_{yy}\,dz = \frac{8(\lambda+\mu)\mu}{\lambda+2\mu} h\,(\sigma_2+\sigma\sigma_1),$$

$$P_{xy} = P_{yx} = \int_{-h}^{h} p_{xy}\,dz = 2\mu h\,.\,\varpi,$$

$$G_{xx} = -G_{yy} = -\int_{-h}^{h} p_{xy} z\,dz = \tfrac{2}{3}\mu h^3 \Pi,$$

$$G_{xy} = \int_{-h}^{h} p_{xx} z\,dz = -\frac{8}{3}\frac{(\lambda+\mu)\mu}{\lambda+2\mu} h^3\,(\Sigma_1+\sigma\Sigma_2),$$

$$G_{yx} = -\int_{-h}^{h} p_{yy} z\,dz = \frac{8}{3}\frac{(\lambda+\mu)\mu}{\lambda+2\mu} h^3\,(\Sigma_2+\sigma\Sigma_1).$$

More accurate expressions would be obtained by multiplying by $(1-zB)$ or by $(1-zA)$, before integration, according as the section considered is perpendicular to x or y; but the error is of the kind already neglected.

3. The second state of strain which we shall consider is one in which, the middle surface being fixed, the displacements are everywhere parallel to it, and given (relative to the same axes as before) by the formulæ

$$\alpha = \frac{3}{4}\frac{P_{xx}}{\mu}\left(\frac{z}{h} - \frac{1}{3}\frac{z^3}{h^3}\right), \quad \beta = \frac{3}{4}\frac{P_{yx}}{\mu}\left(\frac{z}{h} - \frac{1}{3}\frac{z^3}{h^3}\right), \quad \gamma = 0,$$

where P_{xx}, P_{yx} are functions, as yet undetermined, of the position of the point P on the middle surface. We assume, provisionally, that their variations within distances of the order h are small in comparison with the values of the functions themselves. It is easily seen that

$$e_{xx} = \frac{d\alpha}{dx} + A\alpha, \quad e_{yx} = \frac{d\beta}{dx} + B\beta;$$

but the second terms are of the orders hA, hB as compared with the first, and are therefore neglected. Hence

$$e_{xx} = \frac{3}{4}\frac{P_{xx}}{\mu h}\left(1 - \frac{z^2}{h^2}\right), \quad e_{yx} = \frac{3}{4}\frac{P_{yx}}{\mu h}\left(1 - \frac{z^2}{h^2}\right);$$

so that the stress on each face of the shell vanishes. In virtue of the above assumption, the remaining strains are small in comparison with e_{xx}, e_{yz}. It follows that the stresses on normal sections perpendicular to x and y reduce to forces

$$\int_{-h}^{h} \mu e_{xx} dz \quad \text{and} \quad \int_{-h}^{h} \mu e_{yz} dz,$$

or P_{xx}, P_{yz}, respectively, per unit length of the sections.

Although the point is not essential to the present argument, we may remark that, in the end, the strains e_{xx}, e_{yz} appear to be, in general, small compared with those considered in the preceding section, so that the corresponding terms in the expression for the potential energy of the deformation may be neglected. The importance of these strains, in the present synthetic view of the subject, lies in the fact that they do not change sign with z, and that, consequently, the accompanying stresses, when integrated over the thickness, do not vanish as in the former case.

4. The next step is to calculate the resultant force and couple due to the stress on the whole boundary of an element of the shell, in the compound strain which results from the superposition of the two states above described. This force and couple will balance the system of applied forces which, properly distributed through the element, would accurately maintain the strain in question. The equations of equilibrium are then obtained, in accordance with our fundamental principle, by expressing that this system is statically equivalent to the system of forces actually applied to the element.

Let us take rectangular axes of x, y, z fixed anywhere in space, and let us regard the unstrained form of the middle surface as given by equations such as

$$x = f_1(\xi, \eta), \quad y = f_2(\xi, \eta), \quad z = f_3(\xi, \eta),$$

where ξ, η now denote the parameters of the lines of curvature. If we write

$$\frac{1}{h_1^2} = \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2,$$

$$\frac{1}{h_2^2} = \left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2 + \left(\frac{dz}{d\eta}\right)^2,$$

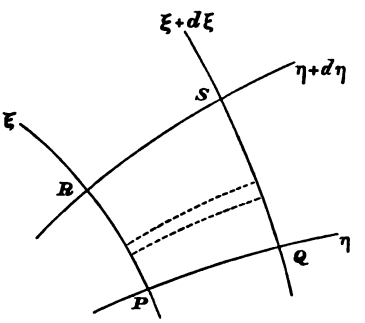
the lengths of elements of the two systems of lines of curvature will be $d\xi/h_1$, $d\eta/h_2$, respectively. It may also be shown that the direction-cosines of the two lines of curvature, and of the normal at a point

$(\xi+d\xi,\ \eta+d\eta)$ of the surface, relative to the lines of curvature and the normal at $(\xi,\ \eta)$, are

$$\begin{matrix} 1, & -h_2\frac{d}{d\eta}\left(\frac{d\xi}{h_1}\right)+h_1\frac{d}{d\xi}\left(\frac{d\eta}{h_2}\right), & \frac{d\xi}{h_1\rho_1}; \\ -h_1\frac{d}{d\xi}\left(\frac{d\eta}{h_2}\right)+h_2\frac{d}{d\eta}\left(\frac{d\xi}{h_1}\right), & 1, & \frac{d\eta}{h_2\rho_2}; \\ -\frac{d\xi}{h_1\rho_1}, & -\frac{d\eta}{h_2\rho_2}, & 1; \end{matrix}$$

respectively, where ρ_1, ρ_2 are the principal radii of curvature, reckoned positive when the corresponding centres of curvature are on the positive side of the normal.

Let us now consider the element of the plate marked out by normals drawn through the contour of the quasi-rectangular area on the middle surface bounded by the lines of curvature whose parameters are ξ, η, $\xi+d\xi$, $\eta+d\eta$; and let the components of the force and couple per unit length of the normal sections through the point (ξ, η) be denoted, in analogy with our former notation, by $P_{\xi\xi}$, $P_{\xi\eta}$, $P_{\eta\xi}$, &c., $G_{\xi\xi}$, $G_{\xi\eta}$, &c.



Resolving parallel to the tangent at P (ξ, η) to the line PQ in the figure, we obtain for the resultant stress in this direction

$$\frac{d}{d\xi}\left(P_{\xi\xi}\cdot\frac{d\eta}{h_2}\right)d\xi+\frac{d}{d\eta}\left(P_{\eta\xi}\cdot\frac{d\xi}{h_1}\right)d\eta+P_{\xi\eta}\frac{d\eta}{h_2}\cdot h_2\frac{d}{d\eta}\left(\frac{d\xi}{h_1}\right)$$
$$-P_{\eta\eta}\frac{d\xi}{h_1}\cdot h_1\frac{d}{d\xi}\left(\frac{d\eta}{h_2}\right)-P_{\xi\eta}\frac{d\eta}{h_2}\cdot\frac{d\xi}{h_1\rho_1};$$

The first two terms arise exactly as in the case of a rectangular element of a plane plate. The third term is found by taking the difference of the resolved parts of the stress $P_{\xi\eta}$ on the two extremities of a strip such as that bounded by the dotted lines in the figure, and adding the results for all the strips obtained by varying η through the range $d\eta$. The fourth term is obtained, in like manner, by dividing the quadrilateral into strips bounded by the lines $\xi = \text{const.}$ The last term is due to the stress $P_{\xi\eta}$ on the section represented by QS. The components of $P_{\eta\xi}$, due to corresponding elements of PQ and RS, *cancel*. If we denote by X, Y, Z the components of applied force

per unit area of the middle surface, parallel to the two lines of curvature and the normal, respectively, we obtain

$$\frac{d}{d\xi}\left(\frac{1}{h_2}P_{\xi\xi}\right)+\frac{d}{d\eta}\left(\frac{1}{h_1}P_{\eta\xi}\right)+P_{\xi\eta}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)-P_{\eta\eta}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)$$
$$-\frac{P_{\xi\xi}}{h_1h_2\rho_1}+\frac{X}{h_1h_2}=0.$$

Similarly, resolving parallel to the tangent line to PR, we find

$$\frac{d}{d\xi}\left(\frac{1}{h_2}P_{\xi\eta}\right)+\frac{d}{d\eta}\left(\frac{1}{h_1}P_{\eta\eta}\right)+P_{\eta\xi}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)-P_{\xi\xi}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)$$
$$-\frac{P_{\eta\xi}}{h_1h_2\rho_2}+\frac{Y}{h_1h_2}=0.$$

The resolution parallel to the normal gives

$$\frac{d}{d\xi}\left(\frac{1}{h_2}P_{\xi\zeta}\right)+\frac{d}{d\eta}\left(\frac{1}{h_1}P_{\eta\zeta}\right)+\frac{P_{\xi\xi}}{h_1h_2\rho_1}+\frac{P_{\eta\eta}}{h_1h_2\rho_2}+\frac{Z}{h_1h_2}=0.$$

In like manner, taking moments about the tangents to PQ, PR, we find

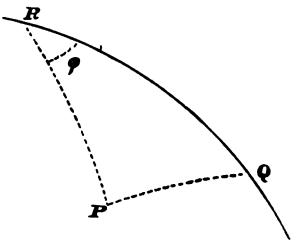
$$\frac{d}{d\xi}\left(\frac{1}{h_2}G_{\xi\xi}\right)+\frac{d}{d\eta}\left(\frac{1}{h_1}G_{\eta\xi}\right)+G_{\xi\eta}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)-G_{\eta\eta}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)$$
$$+\frac{P_{\eta\xi}}{h_1h_2}+\frac{L}{h_1h_2}=0,$$

$$\frac{d}{d\xi}\left(\frac{1}{h_2}G_{\xi\eta}\right)+\frac{d}{d\eta}\left(\frac{1}{h_1}G_{\eta\eta}\right)+G_{\eta\xi}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)-G_{\xi\xi}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)$$
$$-\frac{P_{\xi\xi}}{h_1h_2}+\frac{M}{h_1h_2}=0,$$

where L, M are the impressed couples per unit area. The last two equations enable us to eliminate $P_{\xi\xi}$, $P_{\eta\xi}$, and, if we substitute the values of the remaining stresses given at the end of § 2, we are left with three equations involving the six quantities σ_1, σ_2, ϖ, Σ_1, Σ_2, Π which express the deformation at any point P of the middle surface. In §§ 6, 7, we shall investigate the values of these quantities in terms of the three component displacements of P and of their differential coefficients.

5. We proceed to the boundary conditions. Let X_0, Y_0, Z_0 be the components, parallel to the two lines of curvature and the normal, of the applied force per unit length of the edge, and let L_0, M_0, 0 be the applied couples about the same lines, also reckoned per unit length.

Considering the equilibrium of a triangular element of which two sides are formed by lines of curvature, whilst the hypothenuse is an element ds of the boundary, we find, resolving parallel to the tangent lines at P,



$$P_{\xi\xi}\cos\phi + P_{\eta\xi}\sin\phi - \frac{S\sin\phi}{\rho_1} = X_0,$$

$$P_{\xi\eta}\cos\phi + P_{\eta\eta}\sin\phi + \frac{S\cos\phi}{\rho_2} = Y_0.$$

Here ϕ denotes the angle which the outwardly directed normal to the edge makes with PQ, and S is the shearing force, normal to the middle surface, across any infinitely short line drawn inwards (in any direction) from the edge.* The last terms on the left-hand side are due to the components of this force at Q and R respectively.

Next, taking moments about PQ, PR,

$$G_{\xi\xi}\cos\phi + G_{\eta\xi}\sin\phi - S\cos\phi = L_0,$$

$$G_{\eta\eta}\sin\phi + G_{\xi\eta}\cos\phi - S\sin\phi = M_0.$$

Finally, resolving normal to the middle surface,

$$P_{\xi\zeta}\cos\phi + P_{\eta\zeta}\sin\phi - \frac{dS}{ds} = Z_0.$$

The elimination of S leads to four boundary conditions similar to those of Mr. Love† (pp. 519, 520).

6. It remains to express the six quantities $\sigma_1, \sigma_2, \varpi, \Sigma_1, \Sigma_2, \Pi$, which define the strain of the middle surface, in terms of the component displacements. With this view we construct a three-dimensional system of orthogonal curvilinear coordinates, in the manner explained by Mr. Love; viz., the surfaces $\xi = \text{const.}$, $\eta = \text{const.}$ are the developables generated by the normals to the middle surface along the lines of curvature, and the surfaces $\zeta = \text{const.}$ are the *parallels* to the middle surface, ζ denoting the distance from the latter, measured

* For fuller explanation, see § 5 of the former paper. The shearing force in question is there denoted by ζ.

† Another (but equivalent) form of these conditions is obtained by considering a *rectangular* element, as in the case of a plane plate.

along a normal. The expression for the square of a linear element in this system is

$$dx^2+dy^2+dz^2=\frac{d\xi^2}{h_1^2}+\frac{d\eta^2}{h_2^2}+d\zeta^2,$$

where

$$\frac{1}{h_1^2}=\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2+\left(\frac{dz}{d\xi}\right)^2,$$

$$\frac{1}{h_2^2}=\left(\frac{dx}{d\eta}\right)^2+\left(\frac{dy}{d\eta}\right)^2+\left(\frac{dz}{d\eta}\right)^2.$$

The curvilinear coordinates of a point P of the middle surface being originally $(\xi, \eta, 0)$, let this point receive small displacements u, v, w in the directions of the two lines of curvature, and of the normal, respectively. The coordinates of its new position P' will be $(\xi+h_1u, \eta+h_2v, w)$. In like manner, the coordinates of a point Q on the middle surface, near P, will change from $(\xi+d\xi, \eta+d\eta, 0)$ to

$$(\xi+h_1u+d\xi+d.h_1u,\quad \eta+h_2v+d\eta+d.h_2v,\quad w+dw).$$

Hence, if Q' be the new position of this point, the projections of $P'Q'$ on the principal directions at P' will be

$$\left(\frac{1}{h_1}+\delta.\frac{1}{h_1}\right)(d\xi+d.h_1u),\quad \left(\frac{1}{h_2}+\delta.\frac{1}{h_2}\right)(d\eta+d.h_2v),\quad dw,$$

where $\delta.h_1^{-1}$ denotes the change in the value of $1/h_1$ as we pass from P to P', viz.,

$$\delta.\frac{1}{h_1}=\left(h_1u\frac{d}{d\xi}+h_2v\frac{d}{d\eta}+w\frac{d}{d\zeta}\right).\frac{1}{h_1},$$

and similarly for $\delta.h_2^{-1}$. Hence, neglecting squares and products of u, v, w, &c., the above projections become

$$\frac{d\xi}{h_1}+\left\{h_1u\frac{d}{d\xi}\left(\frac{1}{h_1}\right)+h_2v\frac{d}{d\eta}\left(\frac{1}{h_1}\right)+w\frac{d}{d\zeta}\left(\frac{1}{h_1}\right)+\frac{1}{h_1}\frac{d(h_1u)}{d\xi}\right\}d\xi$$
$$+\frac{1}{h_1}\frac{d(h_1u)}{d\eta}d\eta,$$

$$\frac{d\eta}{h_2}+\frac{1}{h_2}\frac{d(h_2v)}{d\xi}d\xi+\left\{h_1u\frac{d}{d\xi}\left(\frac{1}{h_2}\right)+h_2v\frac{d}{d\eta}\left(\frac{1}{h_2}\right)+w\frac{d}{d\zeta}\left(\frac{1}{h_2}\right)\right.$$
$$\left.+\frac{1}{h_2}\frac{d(h_2v)}{d\eta}\right\}d\eta,$$

$$\frac{dw}{d\xi}d\xi+\frac{dw}{d\eta}d\eta.$$

The first two of these must be equivalent to

$$(1+\sigma_1)\frac{d\xi}{h_1}+(\tfrac{1}{2}\varpi+\omega)\frac{d\eta}{h_2},$$

$$(\tfrac{1}{2}\varpi-\omega)\frac{d\xi}{h_1}+(1+\sigma_2)\frac{d\eta}{h_2},$$

where ω measures the rotation, relative to the substance, of the directions ξ, η as we pass from P to P'. Comparing, we find

$$\sigma_1=\frac{d(h_1u)}{d\xi}+h_1\left(h_1u\frac{d}{d\xi}+h_2v\frac{d}{d\eta}+w\frac{d}{d\zeta}\right)\frac{1}{h_1}$$

$$=h_1\frac{du}{d\xi}+h_1h_2\frac{d}{d\eta}\left(\frac{1}{h_1}\right)v-\frac{w}{\rho_1},$$

$$\sigma_2=h_2\frac{dv}{d\eta}+h_1h_2\frac{d}{d\xi}\left(\frac{1}{h_2}\right)u-\frac{w}{\rho_2},$$

$$\varpi=\frac{h_2}{h_1}\frac{d(h_1u)}{d\eta}+\frac{h_1}{h_2}\frac{d(h_2v)}{d\xi},$$

where we have put

$$\frac{d}{d\zeta}\left(\frac{1}{h_1}\right)=-\frac{1}{h_1\rho_1},\qquad \frac{d}{d\zeta}\left(\frac{1}{h_2}\right)=-\frac{1}{h_2\rho_2},$$

as is evident geometrically.*

7. To calculate Σ_1, Σ_2, Π, we require the projection of $P'Q'$ on the normal to the deformed surface at P' correct to the second order of $d\xi$, $d\eta$. The direction-cosines of this normal, relative to the principal directions at P', are

$$-h_1\frac{dw}{d\xi},\quad -h_2\frac{dw}{d\eta},\quad 1;$$

for we verify at once, on reference to the values given above for the projections of $P'Q'$, that these are the direction-cosines of a line perpendicular to $P'Q'$, for all values of the ratio $d\xi/d\eta$, small quantities of the order of the square of the strain being, as usual, neglected.

Now the direction-cosines of the principal directions at P, relative to the axes of x, y, z, are

$$\left(h_1\frac{dx}{d\xi},\ h_1\frac{dy}{d\xi},\ h_1\frac{dz}{d\xi}\right),\quad \left(h_2\frac{dx}{d\eta},\ h_2\frac{dy}{d\eta},\ h_2\frac{dz}{d\eta}\right),\quad \left(\frac{dx}{d\zeta},\ \frac{dy}{d\zeta},\ \frac{dz}{d\zeta}\right),$$

* These formulæ agree exactly with those given by Mr. Love, p. 506, if we take account of the different convention as to the sign of ρ_1, ρ_2.

respectively. Hence, if R be a point near P whose curvilinear coordinates are $(\xi+D\xi,\ \eta+D\eta,\ D\zeta)$, the projections of PR on the principal directions of P are, to the second order,

$$h_1\frac{dx}{d\xi}\left\{\frac{dx}{d\xi}D\xi+\frac{dx}{d\eta}D\eta+\frac{dx}{d\zeta}D\zeta+\frac{1}{2}\left(\frac{d^2x}{d\xi^2}D\xi^2+\frac{d^2x}{d\eta^2}D\eta^2+\frac{d^2x}{d\zeta^2}D\zeta^2\right.\right.$$
$$\left.\left.+2\frac{d^2x}{d\eta\,d\zeta}D\eta\,D\zeta+2\frac{d^2x}{d\zeta\,d\xi}D\zeta\,D\xi+\frac{d^2x}{d\xi\,d\eta}D\xi\,D\eta\right)\right\}$$
$$+h_1\frac{dy}{d\xi}\left\{\frac{dy}{d\xi}D\xi+\&c.\right\}$$
$$+h_1\frac{dz}{d\xi}\left\{\frac{dz}{d\xi}D\xi+\&c.\right\},$$

with two similar expressions. In virtue of the various relations such as

$$\Sigma\frac{dx}{d\xi}\frac{dx}{d\eta}=0,$$
$$\Sigma\frac{dx}{d\xi}\frac{d^2x}{d\xi^2}=\frac{1}{2}\frac{d}{d\xi}\left(\frac{1}{h_1^2}\right)=\frac{1}{h_1}\frac{d}{d\xi}\left(\frac{1}{h_1}\right),$$
$$\Sigma\frac{dx}{d\zeta}\frac{d^2x}{d\zeta^2}=0,$$
$$\Sigma\frac{dx}{d\xi}\frac{d^2x}{d\eta^2}=-\frac{1}{2}\frac{d}{d\xi}\left(\frac{1}{h_2^2}\right)=-\frac{1}{h_2}\frac{d}{d\xi}\left(\frac{1}{h_2}\right),$$
$$\Sigma\frac{dx}{d\xi}\frac{d^2x}{d\zeta^2}=0,$$
$$\Sigma\frac{dx}{d\zeta}\frac{d^2x}{d\xi^2}=-\frac{1}{2}\frac{d}{d\zeta}\left(\frac{1}{h_1^2}\right)=\frac{1}{h_1^2\rho_1},$$
$$\Sigma\frac{dx}{d\xi}\frac{d^2x}{d\eta\,d\zeta}=0,$$

these reduce to

$$\frac{D\xi}{h_1}+\frac{1}{2}\frac{d}{d\xi}\left(\frac{1}{h_1}\right)D\xi^2-\frac{1}{2}\frac{h_1}{h_2}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)D\eta^2-\frac{1}{h_1^2\rho_1}D\xi D\zeta+\frac{d}{d\eta}\left(\frac{1}{h_1}\right)D\xi D\eta,$$
$$\frac{D\eta}{h_2}-\frac{1}{2}\frac{h_2}{h_1}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)D\xi^2+\frac{1}{2}\frac{d}{d\eta}\left(\frac{1}{h_2}\right)D\eta^2-\frac{1}{h_2^2\rho_2}D\eta D\zeta+\frac{d}{d\xi}\left(\frac{1}{h_2}\right)D\xi D\eta,$$
$$D\zeta+\frac{1}{2}\frac{1}{h_1^2\rho_1}D\xi^2+\frac{1}{2}\frac{1}{h_2^2\rho_2}D\eta^2$$

respectively. In order to deduce from these expressions the projections of $P'Q'$ on the principal directions at P', we must put

$$D\xi = d\xi + d(h_1 u) + \tfrac{1}{2}d^2(h_1 u),$$
$$D\eta = d\eta + d(h_2 v) + \tfrac{1}{2}d^2(h_2 v),$$
$$D\zeta = \quad dw \quad + \tfrac{1}{2}d^2 w,$$

and substitute for h_1, h_2 the values $h_1 + \delta h_1$, $h_2 + \delta h_2$, where

$$\delta = h_1 u \frac{d}{d\xi} + h_2 v \frac{d}{d\eta} + w \frac{d}{d\zeta}.$$

Since the first two projections are afterwards to be multiplied by $-h_1 dw/d\xi$, $-h_2 dw/d\eta$, respectively, we may omit in them all terms depending on u, v, w, which comes to the same thing as writing $d\xi$, $d\eta$, 0 for $D\xi$, $D\eta$, $D\zeta$. We finally obtain, for the required projection on the normal to the deformed surface at P', the value

$$-h_1 \frac{dw}{d\xi}\left\{\frac{1}{2}\frac{d}{d\xi}\left(\frac{1}{h_1}\right)d\xi^2 - \frac{1}{2}\frac{h_1}{h_2}\frac{d}{d\xi}\left(\frac{1}{h_2}\right)d\eta^2 + \frac{d}{d\eta}\left(\frac{1}{h_1}\right)d\xi\,d\eta\right\}$$

$$-h_2 \frac{dw}{d\eta}\left\{-\frac{1}{2}\frac{h_2}{h_1}\frac{d}{d\eta}\left(\frac{1}{h_1}\right)d\xi^2 + \frac{1}{2}\frac{d}{d\eta}\left(\frac{1}{h_2}\right)d\eta^2 + \frac{d}{d\xi}\left(\frac{1}{h_2}\right)d\xi\,d\eta\right\}$$

$$+\frac{1}{2}\left\{\frac{d^2 w}{d\xi^2}d\xi^2 + 2\frac{d^2 w}{d\xi\,d\eta}d\xi\,d\eta + \frac{d^2 w}{d\eta^2}d\eta^2\right\}$$

$$+\frac{1}{2}\frac{1}{h_1^2\rho_1}\left\{\left(1+2\frac{d(h_1 u)}{d\xi}\right)d\xi^2 + 2\frac{d(h_1 u)}{d\eta}d\xi\,d\eta\right\}$$

$$+\frac{1}{2}\frac{1}{h_2^2\rho_2}\left\{2\frac{d(h_2 v)}{d\xi}d\xi\,d\eta + \left(1+2\frac{d(h_2 v)}{d\eta}\right)d\eta^2\right\}$$

$$+\frac{1}{2}d\xi^2\left(h_1 u\frac{d}{d\xi} + h_2 v\frac{d}{d\eta} + w\frac{d}{d\zeta}\right)\frac{1}{h_1^2\rho_1}$$

$$+\frac{1}{2}d\eta^2\left(h_1 u\frac{d}{d\xi} + h_2 v\frac{d}{d\eta} + w\frac{d}{d\zeta}\right)\frac{1}{h_2^2\rho_2}.$$

Equating this to

$$\frac{1}{2}\left\{A_1\frac{d\xi^2}{h_1^2} + 2H_1\frac{d\xi}{h_1}\frac{d\eta}{h_2} + B_1\frac{d\eta^2}{h_2^2}\right\},$$

where A_1, H_1, B_1 have the same meanings as in § 2, we find

$$A_1 = \frac{1}{\rho_1} + h_1^2 \frac{d^2w}{d\xi^2} + h_1 \frac{dh_1}{d\xi}\frac{dw}{d\xi} - \frac{h_2^2}{h_1}\frac{dh_1}{d\eta}\frac{dw}{d\eta}$$

$$+ \frac{2}{\rho_1}\frac{d(h_1u)}{d\xi} + h_1^2\left(h_1u\frac{d}{d\xi} + h_2v\frac{d}{d\eta} + w\frac{d}{d\zeta}\right)\frac{1}{h_1^2\rho_1},$$

$$B_1 = \frac{1}{\rho_2} + h_2^2 \frac{d^2w}{d\eta^2} - \frac{h_1^2}{h_2}\frac{dh_2}{d\xi}\frac{dw}{d\xi} + h_2\frac{dh_2}{d\eta}\frac{dw}{d\eta}$$

$$+ \frac{2}{\rho_2}\frac{d(h_2v)}{d\eta} + h_2^2\left(h_1u\frac{d}{d\xi} + h_2v\frac{d}{d\eta} + w\frac{d}{d\zeta}\right)\frac{1}{h_2^2\rho_2},$$

$$H_1 = h_1h_2\frac{d^2w}{d\xi d\eta} + h_2\frac{dh_1}{d\eta}\frac{dw}{d\xi} + h_1\frac{dh_2}{d\xi}\frac{dw}{d\eta}$$

$$+ \frac{h_2}{h_1\rho_1}\frac{d(h_1u)}{d\eta} + \frac{h_1}{h_2\rho_2}\frac{d(h_2v)}{d\xi}.$$

Remembering that, as shown in § 2,

$$A' = A_1 - 2\sigma_1 A = A_1 - \frac{2\sigma_1}{\rho_1},$$

$$B' = B_1 - 2\sigma_2 B = B_1 - \frac{2\sigma_2}{\rho_2},$$

$$H' = H_1 - \tfrac{1}{2}\varpi(A+B) = H_1 - \tfrac{1}{2}\varpi\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right),$$

and recalling the values of $\sigma_1, \sigma_2, \varpi$, given at the end of §6, we obtain, after a little reduction,

$$\Sigma_1 = A' - A = h_1^2\frac{d^2w}{d\xi^2} + h_1\frac{dh_1}{d\xi}\frac{dw}{d\xi} - \frac{h_2^2}{h_1}\frac{dh_1}{d\eta}\frac{dw}{d\eta}$$

$$+ \left(h_1u\frac{d}{d\xi} + h_2v\frac{d}{d\eta}\right)\frac{1}{\rho_1} + \frac{w}{\rho_1^2},$$

$$\Sigma_2 = B' - B = h_2^2\frac{d^2w}{d\eta^2} - \frac{h_1^2}{h_2}\frac{dh_2}{d\xi}\frac{dw}{d\xi} + h_2\frac{dh_2}{d\eta}\frac{dw}{d\eta}$$

$$+ \left(h_1u\frac{d}{d\xi} + h_2v\frac{d}{d\eta}\right)\frac{1}{\rho_2} + \frac{w}{\rho_2^2},$$

$$\tfrac{1}{2}\Pi = H' = h_1 h_2 \frac{d^2 w}{d\xi d\eta} + h_2 \frac{dh_1}{d\eta}\frac{dw}{d\xi} + h_1 \frac{dh_2}{d\xi}\frac{dw}{d\eta}$$
$$+ \frac{1}{2}\frac{h_2}{h_1}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{d(h_1 u)}{d\eta} + \frac{1}{2}\frac{h_1}{h_2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\frac{d(h_2 v)}{d\xi}.$$ *

We have here made use of the obvious geometrical relations

$$\frac{d\rho_1}{d\zeta} = -1, \quad \frac{d\rho_2}{d\zeta} = -1.$$

8. As a verification of these geometrical formulæ we may take the case of a sphere, writing

$$\xi = \theta, \quad \eta = \omega,$$

$$h_1 = 1/a, \quad h_2 = 1/a \sin\theta, \quad \rho_1 = \rho_2 = -a,$$

where θ, ω are the colatitude and longitude, and the positive direction of the normal is outwards. We find

$$\sigma_1 = \frac{1}{a}\frac{du}{d\theta} + \frac{w}{a},$$

$$\sigma_2 = \frac{1}{a\sin\theta}\frac{dv}{d\omega} + \cot\theta\,\frac{u}{a} + \frac{w}{a},$$

$$\varpi = \frac{1}{a\sin\theta}\frac{du}{d\omega} + \frac{1}{a}\sin\theta\,\frac{d}{d\theta}\left(\frac{v}{\sin\theta}\right),$$

$$\Sigma_1 = \frac{1}{a^2}\frac{d^2 w}{d\theta^2} + \frac{w}{a^2},$$

$$\Sigma_2 = \frac{1}{a^2\sin^2\theta}\frac{d^2 w}{d\omega^2} + \frac{1}{a^2}\cot\theta\frac{dw}{d\theta} + \frac{w}{a^2},$$

$$\tfrac{1}{2}\Pi = \frac{1}{a^2\sin\theta}\frac{d^2 w}{d\theta\, d\omega} - \frac{1}{a^2}\frac{\cos\theta}{\sin^2\theta}\frac{dw}{d\omega}.$$

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* The quantities Σ_1, Σ_2, Π are related to Mr. Love's λ_1, κ_2, κ_1 as follows:—

$$-\lambda_1 = \Sigma_1 + \frac{\sigma_1}{\rho_1}, \quad \kappa_2 = \Sigma_2 + \frac{\sigma_2}{\rho_2}, \quad \kappa_1 = \tfrac{1}{2}\Pi + \tfrac{1}{2}\varpi\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

The differences are practically unimportant, for the reason given in § 1. (There appears to be an error of transcription in the last terms of the expressions for $-\lambda_1$ and κ_2 given by Mr. Love on p. 509 of his memoir. These terms should be

$$-h_1 h_2 v \frac{1}{\rho_2}\frac{d}{d\beta}\left(\frac{1}{h_1}\right) \text{ and } -h_1 h_2 u \frac{1}{\rho_1}\frac{d}{d\alpha}\left(\frac{1}{h_2}\right),$$

respectively.)

We verify that these quantities all vanish when

$$u = C \sin\theta, \quad v = 0, \quad w = -C\cos\theta,$$

which represents a displacement of the shell as a whole parallel to the polar axis; and again when

$$u = 0, \quad v = C\sin\theta, \quad w = 0,$$

which corresponds to a rotation. As a further test, let us suppose that $u = 0$, $v = 0$, whilst $w = S_n$, a spherical harmonic of order n. We find, for the change in the sum of the principal curvatures,

$$\Sigma_1 + \Sigma_2 = \frac{1}{a^2}\left\{\frac{d^2S_n}{d\theta^2} + \cot\theta\frac{dS_n}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2S_n}{d\omega^2} + 2S_n\right\}$$

$$= -\frac{1}{a^2}(n-1)(n+2)\,S_n,$$

by the differential equation of S_n. This is a known result.*

9. Let us now consider more particularly the case of a cylindrical shell. The axis of x being coincident with that of the cylinder, we may write

$$\xi = x, \quad \eta = \omega, \quad h_1 = 1, \quad h_2 = 1/a, \quad \rho_1 = \infty, \quad \rho_2 = -a,$$

where ω is the angular coordinate, and the positive direction of the normal is outwards. These give

$$\sigma_1 = \frac{du}{dx}, \quad \sigma_2 = \frac{dv}{ad\omega} + \frac{w}{a}, \quad \varpi = \frac{du}{ad\omega} + \frac{dv}{dx},$$

$$\Sigma_1 = \frac{d^2w}{dx^2}, \quad \Sigma_2 = \frac{d^2w}{a^2d\omega^2} + \frac{w}{a^2}, \quad \tfrac{1}{2}\Pi = \frac{d^2w}{ad\omega\,dx} + \frac{1}{2a^2}\frac{du}{d\omega} - \frac{1}{2a}\frac{dv}{dx}.\dagger$$

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* The formulæ given by Mr. Love (p. 512) for the principal curvatures do not appear to verify in this case. ["These formulæ are known to be incorrect. There are mistakes in the working. On p. 511, *for* $E = \frac{1-2\sigma_1}{h_1^2}$, *read* $E = \frac{1+2\sigma_1}{h_1^2}$, &c.; and *for* $E' = -\frac{1+\sigma_1}{h_1^3 h_2}\lambda_1'$, *read* $E' = -\frac{(1+\sigma_1)^2(1+\sigma_2)}{h_1^3 h_2}\lambda_1'$, &c. This makes all right."—*Note by Referee.*]

† We verify that these quantities all vanish when

$$u = -Ca\sin\omega, \quad v = Cx\cos\omega, \quad w = Cx\sin\omega,$$

which corresponds to a rotation of the shell as a whole about the diameter for which $x = 0$, $\omega = 0$.

The equations of § 4 then become

$$\left.\begin{aligned} a\frac{dP_{xx}}{dx}+\frac{dP_{x\omega}}{d\omega}+aX&=0,\\ a\frac{dP_{x\omega}}{dx}+\frac{dP_{\omega\omega}}{d\omega}+P_{\omega\chi}+aY&=0,\\ a\frac{dP_{x\chi}}{dx}+\frac{dP_{\omega\chi}}{d\omega}-P_{\omega\omega}+aZ&=0, \end{aligned}\right\}$$

where (if for simplicity we suppose the impressed couples L, M to vanish) the shearing forces $P_{x\chi}$, $P_{\omega\chi}$ are given by

$$\left.\begin{aligned} aP_{x\chi}&= a\frac{dG_{x\omega}}{dx}+\frac{dG_{\omega\omega}}{d\omega},\\ aP_{\omega\chi}&=-a\frac{dG_{xx}}{dx}-\frac{dG_{\omega x}}{d\omega}. \end{aligned}\right\}$$

At a free edge coincident with a circular section we have $\phi=0$ or π, so that the boundary conditions of § 5 reduce to

$$\left.\begin{aligned} P_{xx}=0,\quad P_{x\omega}-\frac{S}{a}&=0,\\ G_{xx}-S=0,\quad G_{x\omega}&=0,\\ P_{x\chi}-\frac{dS}{ds}&=0. \end{aligned}\right\}$$

10. As an application of these equations, we may investigate the two-dimensional vibrations of an infinitely long cylinder. In this case we have $u=0$, and all differential coefficients with respect to x vanish, so that

$$\sigma_1=0,\quad \varpi=0,\quad \Sigma_1=0,\quad \Pi=0,$$

whilst

$$\sigma_2=\frac{1}{a}\left(\frac{dv}{d\omega}+w\right),$$

$$\Sigma_2=\frac{1}{a^2}\left(\frac{d^2w}{d\omega^2}+w\right).$$

Hence $P_{\omega\omega}=0$, $G_{xx}=G_{\omega\omega}=0$, and the equations further reduce to

$$\left.\begin{aligned} \frac{dP_{\omega\omega}}{d\omega}+P_{\omega\chi}+aX&=0,\\ \frac{dP_{\omega\chi}}{d\omega}-P_{\omega\omega}+aY&=\ , \end{aligned}\right\}$$

where $$P_{\omega\omega} = -\frac{1}{a}\frac{dG_{\omega r}}{d\omega}.$$

Assuming $$v = B\sin s\omega, \quad w = C\cos s\omega,$$

where s is integral, we find, by § 2,

$$P_{\omega\omega} = 6A\frac{h}{a}(sB+C)\cos s\omega,$$

$$P_{\omega r} = -2A\frac{h^3}{a^3}s(s^2-1)C\sin s\omega,$$

where $$A = \frac{4}{3}\frac{(\lambda+\mu)\mu}{\lambda+2\mu}.$$

To form the equations of motion, we write

$$Y = -2h\rho\frac{d^2v}{dt^2}, \quad Z = -2h\rho\frac{d^2w}{dt^2},$$

where ρ is the density. Assuming that all our functions vary as e^{ipt}, and putting

$$p^2\rho a^2/A = \beta^2,$$

we get

$$\left.\begin{aligned}(\beta^2-3s^2)B - s\left\{3+(s^2-1)\frac{h^2}{a^2}\right\}C = 0,\\ -3sB+\left\{\beta^2-3-s^2(s^2-1)\frac{h^2}{a^2}\right\}C = 0,\end{aligned}\right\}$$

whence

$$(\beta^2-3s^2)\left\{\beta^2-3-s^2(s^2-1)\frac{h^2}{a^2}\right\} - 3s^2\left\{3+(s^2-1)\frac{h^2}{a^2}\right\} = 0.$$

Since h/a is small, the two roots of this quadratic in β^2 are

$$\beta^2 = 3(s^2+1),$$

and $$\beta^2 = \frac{s^2(s^2-1)^2}{s^2+1}\frac{h^2}{a^2},$$

approximately. These are both known results. The modes of vibration corresponding to the former root belong to the class of cases to which special attention has been paid by Mr. Love, where the dominant feature is the extension of the middle surface, and the frequency is, accordingly, independent of the thickness. They can of course be investigated by a much simpler analysis, if we recognise

this peculiarity from the outset. The second root belongs to the well-known flexural vibrations of a cylinder,* and gives a frequency proportional to the thickness. With this value of β^2 we find

$$\frac{sB}{C} = -1 - \frac{2}{3}\frac{s^2(s^2-1)}{s^2+1}\frac{h^2}{a^2},$$

whence

$$\sigma_2 = -\frac{2}{3}\frac{s^2(s^2-1)}{s^2+1}\frac{h^2}{a^2}\cdot\frac{C}{a}\cos s\omega,$$

where C is the maximum amplitude in the direction of the normal. This is of the order h/a as compared with the maximum strain ($h\Sigma_2$) due to the flexure.

11. The full discussion (from our present standpoint) of the free vibrations of an open cylinder of *finite* length would lead to some very long equations. The effect of a free edge† may, however, be sufficiently illustrated by the following statical problem, which admits of very simple treatment.

Consider a shell in the form of a portion of a cylinder bounded by two generators and two circular arcs (for example, a boiler plate), and let us suppose that forces are applied to the straight edges so that the strained form shall still be a surface of revolution, the curved edges being free. Under these circumstances, u and w will be functions of x only, whilst

$$v = C\omega,$$

this being the only form which makes $\varpi = 0$, and σ_2 independent of ω. Hence

$$\sigma_1 = \frac{du}{dx}, \quad \sigma_2 = \frac{C+w}{a}, \quad \varpi = 0,$$

$$\Sigma_1 = \frac{d^2w}{dx^2}, \quad \Sigma_2 = \frac{w}{a^2}, \quad \Pi = 0.$$

These make

$$P_{xx} = 0, \quad G_{xz} = 0, \quad G_{zz} = 0,$$

and thence

$$P_{xx} = \frac{dG_{xx}}{dx}, \quad P_{zx} = 0.$$

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* Rayleigh, *Sound*, § 233.

† The reader who is interested in this point, but is not attracted by the general investigations of §§ 1–7, may easily construct for himself elementary proofs of the formulæ for the particular case considered in this section.

The general equations of § 9 then reduce to

$$\left.\begin{aligned} \frac{dP_{xx}}{dx} &= 0, \\ a\frac{dP_{xx}}{dx} - P_{\omega\omega} &= 0, \end{aligned}\right\}$$

and the conditions at the circular edges ($x = \pm l$) to

$$P_{xx} = 0, \quad G_{x\omega} = 0, \quad P_{x\chi} = 0.$$

It appears that $P_{xx} = 0$ everywhere, and therefore

$$\sigma_1 + \sigma\sigma_2 = 0.$$

Using this to simplify the expression for $P_{\omega\omega}$, the second general equation becomes

$$\frac{h^2}{3}\frac{d^2}{dx^2}\left\{\frac{d^2w}{dx^2} + \sigma\frac{w}{a^2}\right\} + (1-\sigma^2)\frac{w+C}{a^2} = 0.$$

The solution of this is

$$w = -C + \Sigma D e^{mx/a},$$

provided $$m^4 + \sigma m^2 + 3(1-\sigma^2)\frac{a^2}{h^2} = 0.$$

Since a/h is large, the roots of this are

$$m = \pm (1 \pm i)\, q,$$

approximately, where

$$q^2 = \frac{\sqrt{3}}{2}(1-\sigma^2)^{\frac{1}{2}} \cdot \frac{a}{h}.$$

Hence, retaining only even functions of x,

$$w = -C + D_1 \cosh\frac{qx}{a}\cos\frac{qx}{a} + D_2 \sinh\frac{qx}{a}\sin\frac{qx}{a}.$$

We have still to satisfy the boundary conditions

$$\left.\begin{aligned} \frac{d^2w}{dx^2} + \sigma\frac{w}{a^2} &= 0, \\ \frac{d}{dx}\left(\frac{d^2w}{dx^2} + \sigma\frac{w}{a^2}\right) &= 0. \end{aligned}\right\}$$

On account of the greatness of q, the second terms in these may be

replaced by $-\sigma C/a^2$, and zero, respectively. We find

$$-2D_1 \sinh \frac{ql}{a} \sin \frac{ql}{a} + 2D_2 \cosh \frac{ql}{a} \cos \frac{ql}{a} = \frac{\sigma C}{q^2},$$

$$-D_1 \left(\cosh \frac{ql}{a} \sin \frac{ql}{a} + \sinh \frac{ql}{a} \cos \frac{ql}{a} \right)$$

$$+ D_2 \left(\sinh \frac{ql}{a} \cos \frac{ql}{a} - \cosh \frac{ql}{a} \sin \frac{ql}{a} \right) = 0,$$

whence

$$D_1 = \frac{\sigma C}{q^2} \cdot \frac{\sinh \frac{ql}{a} \cos \frac{ql}{a} - \cosh \frac{ql}{a} \sin \frac{ql}{a}}{\sinh \frac{2ql}{a} + \sin \frac{2ql}{a}},$$

$$D_2 = \frac{\sigma C}{q^2} \cdot \frac{\cosh \frac{ql}{a} \sin \frac{ql}{a} + \sinh \frac{ql}{a} \cos \frac{ql}{a}}{\sinh \frac{2ql}{a} + \sin \frac{2ql}{a}}.$$

This constitutes the complete solution of our problem.*

It appears that the deviation of the strained surface from the cylindrical form given by $w = -C$ is, at most, of the order h/a; but that the extension σ_2, $= (w + C)/a$, may in the neighbourhood of the edge become of the same order as the maximum strain ($h\Sigma_2$, $= -hC/a^2$, approximately), due to the flexure. Something here turns on the value of ql/a, *i.e.*, on the ratio which the breadth of the plate bears to the mean proportional between the radius and the thickness. Let us first suppose this ratio to be large. We then have

$$D_1 = \frac{\sigma C}{q^2} e^{-ql/a} \left(\cos \frac{ql}{a} - \sin \frac{ql}{a} \right),$$

$$D_2 = \frac{\sigma C}{q^2} e^{-ql/a} \left(\cos \frac{ql}{a} + \sin \frac{ql}{a} \right).$$

These are very small compared with C, and may be neglected except near the edges, where they are (in the values of w, &c.) multiplied by the large factors $\cosh qx/a$, $\sinh qx/a$. Near the edge $x = l$ we have.

* Exactly the same result follows from Mr. Love's forms of the fundamental equations.

for example,

$$w = -C + \frac{1}{\sqrt{2}} \frac{\sigma C}{q^2} e^{-q(l-x)/a} \cos\left(q\frac{l-x}{a} + \frac{\pi}{4}\right).$$

The deviation from the cylindrical form has for its maximum value

$$\frac{\sigma}{\sqrt{3}.(1-\sigma^2)^{\frac{1}{2}}} \cdot \frac{h}{a} \cdot C,$$

and becomes insensible at a distance from the edge comparable with the mean proportional aforesaid. Since

$$\sigma_2 = (C+w)/a, \quad \sigma_1 = -\sigma\sigma_2,$$

the preceding formula also gives the amount and the distribution of the extensions σ_1, σ_2. Since these are, at the circular edges, of the same order as $h\Sigma_1$, $h\Sigma_2$, it becomes of interest to compare the two parts of the potential energy which depend on the extension and the flexure, respectively. If, as before, we write

$$A = \frac{4}{3} \frac{(\lambda+\mu)\mu}{\lambda+2\mu},$$

the former part of the energy is, per unit length of the circular section,

$$= 3Ah \int_{-l}^{l} (\sigma_1^2 + 2\sigma\sigma_1\sigma_2 + \sigma_2^2)\, dx$$

$$= 3A(1-\sigma^2) \frac{h}{a^2} \int_{-l}^{l} (w+C)^2\, dx.$$

Substituting the above approximate value of w, we obtain

$$3A\sigma^2(1-\sigma^2) \frac{hC^2}{q^4a^2} \int^{l} e^{-2q(l-x)/a} \cos^2\left(q\frac{l-x}{a} + \frac{\pi}{4}\right) dx,$$

where the integration now extends over the short distance from the edge $x = l$ within which the exponential factor differs sensibly from zero. The result is

$$\tfrac{3}{8}A\sigma^2(1-\sigma^2) \frac{hC^2}{q^5a}.$$

The flexural part of the energy is

$$Ah^3 \int_{-l}^{l} (\Sigma_1^2 + 2\sigma\Sigma_1\Sigma_2 + \Sigma_2^2)\, dx.$$

The most important term in this is obtained by putting

$$\Sigma_2 = -C/a^2, \quad \Sigma_1 = 0,$$

and is $$2\frac{Ah^3C^2}{a^4}l.$$

The ratio of the former part of the energy to the latter is of the order $\sqrt{(ah)}/l$, which is, by hypothesis, small. This tends, I think, to remove the difficulty raised by Mr. Love, by showing that, although the presence of a free edge does in fact give rise to extensional strains comparable with those due to the flexure, the area to which they are practically confined is relatively so small that their contribution to the total energy is insignificant.

The distribution of applied force over the straight edges of the plate is somewhat artificial, but it may without sensible error be replaced by a much simpler one. It consists of a couple having the edge as axis, given by

$$G_{xx} = 2Ah^3(\Sigma_2 + \sigma\Sigma_1) = 2A\frac{h^3}{a^2}\left(w + \sigma a^2\frac{d^2w}{dx^2}\right),$$

and of a tension perpendicular to the edge, of amount

$$P_{xx} = 6Ah(\sigma_2 + \sigma\sigma_1) = 6A(1-\sigma^2)\frac{h}{a}(C+w).$$

Making use of the approximate formula for w, we find that near the edge $x = l$,

$$G_{xx} = 2A\frac{h^3}{a^2}C\left\{-1 + \sqrt{2}\,\sigma^2 e^{-q(l-x)/a}\sin\left(q\frac{l-x}{a} + \frac{\pi}{4}\right)\right\},$$

$$P_{xx} = 3\sqrt{2}A\sigma(1-\sigma^2)\frac{hC}{q^2a}e^{-q(l-x)/a}\cos\left(q\frac{l-x}{a} + \frac{\pi}{4}\right).$$

The deviations of G_{xx} from uniformity, and of P_{xx} from zero, are confined to a short distance from the corner. The integral of P_{xx} over this distance is readily found to be zero, as might have been foreseen from the fact that the plate as a whole would not otherwise be in equilibrium. The integral of the variable part of G_{xx} is of the order $\sqrt{(ah)}/l$, as compared with that of the constant part taken over the whole length of the edge. The theory of "local perturbations,"* therefore, warrants us in asserting that, if the tension and the non-uniform part of the couple be suppressed, no appreciable change will take place except in the immediate neighbourhood of the corners. Hence, subject to this qualification, we have solved the problem of the flexure of a cylindrical plate by a pair of equal and opposite couples applied uniformly over the straight edges.

* The term is adopted from Boussinesq, who has treated this subject very fully. See his "Application des Potentiels à l'Étude de l'Equilibre et du Mouvement des Solides Elastiques," Paris, 1885.

When, on the other hand, $\sqrt{(ah)}$ is comparable with l, we must have recourse to the more accurate values of D_1, D_2. The value of σ_2 at the circular edges is found to be

$$\frac{\sigma C}{2q^2a}\cdot\frac{\sinh\frac{2ql}{a}-\sin\frac{2ql}{a}}{\sinh\frac{2ql}{a}+\sin\frac{2ql}{a}}.$$

As l is diminished, this tends ultimately to the value $\sigma Cl^2/3a^2$, which is of the order l^2/ah compared with the maximum strain due to flexure. The flexural rigidity of the shell—*i.e.*, the ratio of the bending couple to the change of curvature—is

$$\int_{-l}^{l} G_{zz}\,dx/(-C/a^2) = 4Ah^2l\left\{1-\frac{\sigma^2a}{ql}\,\frac{\cosh\frac{2ql}{a}-\cos\frac{2ql}{a}}{\sinh\frac{2ql}{a}+\sin\frac{2ql}{a}}\right\}.$$

When ql/a is small, this reduces to

$$4A(1-\sigma^2)\,h^2l,$$

which is, in fact, the flexural rigidity of a *bar* whose section is a rectangle of breadth $2l$ and depth $2h$, since

$$A(1-\sigma^2) = \tfrac{1}{3}E.$$

where E is Young's modulus.

[*Note added Feb.* 15, 1890.—The paper as read contained (in the form of an Appendix) an investigation, based on the general equations of elasticity, of the uniform flexure of an infinitely long cylindrical plate of finite thickness. This is now withdrawn, the problem in question having been solved in a paper by Lord Rayleigh,* which has since appeared. The most important result is the calculation of the stress across any element of area parallel to the middle surface. When the ratio h/a is small, this was found to be

$$p_{rr} = -2\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\cdot\frac{R_0}{a}\cdot\frac{z^2-h^2}{a^2},$$

approximately,† where $a+R_0$ is the radius of the middle surface in

* *Proceedings*, June 13, 1889, p. 372.
† *Cf.* equations (53) and (57) of the paper referred to.

the strained state. This formula may be obtained more simply, if we accept the state of strain described in § 1, above, as a first approximation. Using polar coordinates r, θ, we have

$$\frac{d}{dr}(p_{rr} \cdot r) = p_{\theta\theta}.$$

Writing $r = a+z$, $p_{\theta\theta} = Kz/a$, integrating, and expressing that p_{rr} vanishes for $z = \pm h$, we obtain

$$p_{rr}(a+z) = \frac{K}{2a}(z^2-h^2),$$

or, to the second order of h/a,

$$p_{rr} = \tfrac{1}{2}K\frac{z^2-h^2}{a^2},$$

which is the result in question, since, on reference to p. 125, above, we find

$$K = -\frac{4(\lambda+\mu)\mu}{\lambda+2\mu} \cdot \frac{R_0}{a}.$$

The first part of Lord Rayleigh's paper contains, for a particular case, an examination of the effects of different modes of application of the same force (as estimated per unit area of the middle surface) to an infinitely long cylindrical shell. It appears* that the resulting variations in the potential energy are of the order h^2/a^2 as compared with the total amount. This may be regarded as a confirmation, to a certain extent, of the fundamental assumption of this paper.]

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* Equation (40). A somewhat similar investigation for the case of a plane plate is given in the *Proceedings*, Dec. 12, 1889, pp. 86–90.

On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points. By A. B. KEMPE, M.A., F.R.S.

[*Read Jan. 9th*, 1890.]

INTRODUCTORY.

Scope of the Paper.

1. In my "Memoir on the Theory of Mathematical Form" (*Phil. Trans.*, Vol. CLXXVII., p. 1), and the Note thereon (*Proc. Roy. Soc.*, Vol. XLII., p. 193), I enunciated the principle that, so far as the processes of exact thought are concerned, the properties of any subject-matter depend solely on the fact that it possesses "form"—*i.e.*, that it consists of a number of entities, certain individuals, pairs, triads, &c. of which are exactly like each other in all their relations, and certain not; these like and unlike individuals, pairs, triads, &c. being distributed through the whole system of entities in a definite way. In the present paper, which is complete in itself, I first consider and fully discuss the properties of a special system of entities, here called the *base system*, the form of which is fully determined by laws which define the mode of distribution of certain like and unlike triads of its component entities—*i.e.*, laws that specify which triads are like and which unlike. I then show that this system is precisely that which is under consideration by the logician, when discussing classes and their mutual relations. I further show that certain entities of the base system, selected in accordance with a simple law, compose a system which is precisely that which is under consideration by the geometrician when discussing points and their mutual relations. I have, in addition, inserted some paragraphs showing that the base system is also that which is under consideration by the logician when dealing with statements and their mutual relations; and have at the end of the paper pointed out that the results obtained suggest an important extension of the ordinary geometrical theory of points.

I should state that, though I have in the Memoir beforementioned defined the form of the system composed by classes, that definition is not of so complete and satisfactory a character as the one now given, which, like the rest of the paper, is, I believe, entirely new.

The Base System.

Definition.

2. I consider a system of an infinite number of entities $a, b, c, d, \ldots$ which are all exactly alike,—are *undistinguished* from each other, so that whatever relation one of them a bears to any collection of them $p, q, r, s, \ldots$, that relation is borne by any other of them b to some collection of them $u, v, w, x, \ldots$ which is undistinguished from $p, q, r, s, \ldots$.

3. Each entity a is accompanied by an infinite number of *equivalent* entities; *i.e.*, entities which are such that, whatever relation one of them bears to any collection of entities of the system $p, q, r, s, \ldots$, that relation is borne by each of the others to the same collection $p, q, r, s, \ldots$. The equivalent entities may be represented by like letters $a, a, a, \ldots$; or we may represent them by different letters and denote their equivalence by an equation, thus: $a = b$.

4. Triads of the entities $a, b, c, \ldots$ are of two sorts, which I shall term, respectively, *linear* and *non-linear*. Except in one case to which I will presently refer, linear triads are unsymmetrical, each consisting of two entities, which I will call the *even* members of the triad, and one which I will call the *odd* member. I shall represent the linear triad, of which a and b are the even members and c is the odd member, by the symbol $ab{\cdot}c$. I shall also use the expression "we have $ab{\cdot}c$" as a convenient abbreviation for "a, b, c compose a linear triad in which a and b are the even members and c the odd member," and similar shortened expressions will be used in other cases which will explain themselves. The exception to the general rule that linear triads are unsymmetrical occurs in the case in which the two even members are equivalent, in which case (see § 5, Law III.) all three members of the triad are equivalent, and therefore the triad is symmetrical.

5. The distribution of the linear triads through the system is defined by the following laws:—

Law I.—If we have $ap{\cdot}b$ and $cp{\cdot}d$, q exists such that we have $ad{\cdot}q$ and $bc{\cdot}q$.

Law II.—If we have $ab{\cdot}p$ and $cp{\cdot}d$, q exists such that we have $aq{\cdot}d$ and $bc{\cdot}q$.

Law III.—If we have $ab \cdot c$, and $a = b$, then $c = a = b$.

Law IV.—If $a = b$, then we have $ac \cdot b$ and $bc \cdot a$, whatever entity of the system c may be.

6. To these laws should be added, for the complete definition of the system, the following *law of continuity*:—

Law V.—No entity is absent from the system which can consistently be present.

Thus it is consistent with the laws of § 5 that there should be entities making linear triads with any two given entities of the system; that, if a, b, c are any three entities of the system, there should be an entity x such that we have $ab \cdot x$, $bc \cdot x$, and $ca \cdot x$; and so on. All these entities are to be supposed to be contained in the system, and are not arbitrarily to be supposed to be absent.

The system is now fully defined. I proceed to consider its properties.

Elementary Properties.

7. If we have $ac \cdot b$ and $bc \cdot a$, then $a = b$. For, since we have $ac \cdot b$ and $bc \cdot a$, by Law I. q exists such that we have $aa \cdot q$ and $bb \cdot q$. And, since we have $aa \cdot q$ and $bb \cdot q$, by Law III. $a = q$ and $b = q$, and therefore $a = b$.

8. If we have $ab \cdot c$ and $bc \cdot d$, we also have $ab \cdot d$ and $ad \cdot c$. For, since we have $ab \cdot c$ and $bc \cdot d$, by Law II. q exists such that we have $aq \cdot d$ and $bb \cdot q$. And, since we have $bb \cdot q$, by Law III. we have $b = q$, and therefore we have $ab \cdot d$.

Again, since we have $ab \cdot c$ and $bc \cdot d$, by Law I. r exists such that we have $ad \cdot r$ and $cc \cdot r$. And, since we have $cc \cdot r$, by Law III. we have $c = r$, and therefore we have $ad \cdot c$.

9. If we have $ab \cdot c$, $ab \cdot d$, and $cd \cdot e$, we have $ab \cdot e$. For, since we have $ab \cdot c$ and $cd \cdot e$, by Law II. p exists such that we have $ap \cdot e$ and $bd \cdot p$. Also, since we have $bd \cdot p$ and $ab \cdot d$, by § 8 we have $ab \cdot p$. Thus, since we have $ap \cdot e$ and $ab \cdot p$, by § 8 we have $ab \cdot e$.

10. If we have $ab \cdot d$, $ac \cdot e$, and $de \cdot b$, we have $cd \cdot b$. For, since we have $ab \cdot d$ and $ac \cdot e$, v exists such that we have $bc \cdot v$ and $cd \cdot v$ (Law I.). Also, since we have $be \cdot v$ and $de \cdot b$, we have $dv \cdot b$ (§ 8). Therefore, since we have $dv \cdot b$ and $cd \cdot v$, we have $cd \cdot b$ (§ 8).

11. If we have $ab \cdot d$, $ac \cdot d$, and $bc \cdot e$, we have $ae \cdot d$. For, since we

7'. If $ab \cdot c$, then $ba \cdot c$

have $ac \cdot d$ and $bc \cdot e$, w exists such that we have $ae \cdot w$ and $bd \cdot w$ (Law I.). Also, since we have $bd \cdot w$ and $ab \cdot d$, we have $aw \cdot d$ (§ 8). Therefore, since we have $aw \cdot d$ and $ae \cdot w$, we have $ae \cdot d$ (§ 8).

Symmetrical Resultants.

12. If we have the six linear triads $\begin{pmatrix} ab \cdot x \\ bc \cdot x \\ ca \cdot x \end{pmatrix}$ and $\begin{pmatrix} ab \cdot y \\ bc \cdot y \\ ca \cdot y \end{pmatrix}$, then $x = y$.

For we have $\begin{pmatrix} ab \cdot x \\ cb \cdot x \\ ac \cdot y \end{pmatrix}$, and therefore $by \cdot x$ (§ 11),

And we have $\begin{pmatrix} ab \cdot y \\ cb \cdot y \\ ac \cdot x \end{pmatrix}$, and therefore $bx \cdot y$ (§ 11),

Thus we have $\begin{pmatrix} by \cdot x \\ bx \cdot y \end{pmatrix}$, and therefore $x = y$ (§ 7).

13. It follows from this that x bears an unique relation to the triad a, b, c. It may therefore be represented by $[abc]$, where

$$[abc] = [acb] = [bca] = [bac] = [cab] = [cba].$$

I shall call $[abc]$ the *symmetrical resultant* of a, b, c.

14. We have $[aac] = a$. For, if $x = [aac]$, we have $aa \cdot x$, and therefore $x = a$ (Law III.).

15. If $[abc] = a$, we have $bc \cdot a$; and if we have $bc \cdot a$, we have $[abc] = a$. For, if $[abc] = a$, we have $bc \cdot a$ by the definition of $[abc]$; and if we have $bc \cdot a$, since we also have $ab \cdot a$ and $ac \cdot a$ (Law IV.), we have $[abc] = a$.

16. We have $\big[ab\,[cde]\big] = \big[[abc][abd]\,e\big]$.

Let $[cde] = h$, $[abh] = x$, $[abc] = k$, $[abd] = l$, $[kle] = y$.

Then we have

$$\begin{pmatrix} cd \cdot h \\ de \cdot h \\ ec \cdot h \end{pmatrix}, \quad \begin{pmatrix} ab \cdot x \\ bh \cdot x \\ ha \cdot x \end{pmatrix}, \quad \begin{pmatrix} ab \cdot k \\ bc \cdot k \\ ca \cdot k \end{pmatrix}, \quad \begin{pmatrix} ab \cdot l \\ bd \cdot l \\ da \cdot l \end{pmatrix}, \quad \begin{pmatrix} kl \cdot y \\ lc \cdot y \\ ek \cdot y \end{pmatrix};$$

and I have to show that $x = y$.

We have $\begin{pmatrix} cb\cdot k \\ ca\cdot k \\ ab\cdot x \end{pmatrix}$, and therefore also $cx\cdot k$ (§ 11).

We have $\begin{pmatrix} db\cdot l \\ da\cdot l \\ ab\cdot x \end{pmatrix}$, „ „ $dx\cdot l$ „

We have $\begin{pmatrix} hb\cdot x \\ ha\cdot x \\ ab\cdot k \end{pmatrix}$, „ „ $hk\cdot x$ „

We have $\begin{pmatrix} hb\cdot x \\ ha\cdot x \\ ab\cdot l \end{pmatrix}$, „ „ $hl\cdot x$ „

We have $\begin{pmatrix} cx\cdot k \\ ce\cdot h \\ hk\cdot x \end{pmatrix}$, „ „ $ek\cdot x$ (§ 10).

We have $\begin{pmatrix} dx\cdot l \\ de\cdot h \\ hl\cdot x \end{pmatrix}$, „ „ $el\cdot x$ „

We have $\begin{pmatrix} cx\cdot k \\ cd\cdot h \\ hk\cdot x \end{pmatrix}$, „ „ $dk\cdot x$ „

We have $\begin{pmatrix} dk\cdot x \\ dx\cdot l \end{pmatrix}$, „ „ $kl\cdot x$ (§ 8).

We have therefore $\begin{pmatrix} kl\cdot x \\ le\cdot x \\ ek\cdot x \end{pmatrix}$; and we also have $\begin{pmatrix} kl\cdot y \\ le\cdot y \\ ek\cdot y \end{pmatrix}$.

Hence $x = y$ (§ 12).

17. We have

$$[ab\,[cde]] = [[abc][abd][abe]].$$

For
$$\begin{aligned}
[ab\,[cde]] &= [c\,[abe][abd]] && (\S\,16)\\
&= \Big[[ca\,[abe]]\,[cb\,[abe]]\,d\Big] && \text{„}\\
&= \Big[[[caa][cae]b]\,[[cbb][cbe]a]\,d\Big] && \text{„}\\
&= \Big[[[cae]ab]\,[[cbe]ab]\,d\Big] && (\S\,14)\\
&= \Big[[[cab][eab]a]\,[[cab][eab]b]\,d\Big] && (\S\,16)\\
&= [[cab][eab][dab]] && \text{„}
\end{aligned}$$

18. We have

$$[ab\,[acd]] = [ad\,[abc]] = [ac\,[abd]].$$

For $[ab\,[acd]] = [[aba][abc]\,d]$ (§ 16)

$= [a\,[abc]\,d]$ (§ 14)

$= [[aad][abd]\,c]$ (§ 16)

$= [a\,[abd]\,c]$ (§ 14).

19. We have $[ab\,[abc]] = [abc]$.

For $[ab\,[abc]] = [[abu][abb]\,c]$ (§ 16)

$= [abc]$ (§ 14).

Unsymmetrical Resultants.

20. If we have $\begin{pmatrix} ab\cdot m \\ cm\cdot a \\ cm\cdot b \end{pmatrix}$ and $\begin{pmatrix} ab\cdot n \\ cn\cdot a \\ cn\cdot b \end{pmatrix}$, then $m = n$.

For we have $\begin{pmatrix} mc\cdot a \\ mc\cdot b \\ ab\cdot n \end{pmatrix}$, and therefore also $mc\cdot n$ (§ 9).

And we have $\begin{pmatrix} nc\cdot a \\ nc\cdot b \\ ab\cdot m \end{pmatrix}$, „ „ $nc\cdot m$ „

Thus we have $\begin{pmatrix} nc\cdot m \\ mc\cdot n \end{pmatrix}$, „ „ $m = n$ (§ 7).

21. It follows from this that m bears an unique relation to the triad a, b, c. It may therefore be represented by $\{ab, c\}$, where $\{ab, c\} = \{ba, c\}$. I shall call $\{ab, c\}$ the *unsymmetrical resultant* of a, b, c.

22. We have $\{aa, b\} = a$. For, if $\{aa, b\} = m$, we have $aa\cdot m$, and thus $m = a$ (Law III.).

23. We have $\{ab, a\} = b$. For, if $\{ab, a\} = n$, we have $\begin{pmatrix} na\cdot b \\ ab\cdot n \end{pmatrix}$, and thus $n = b$ (§ 7).

24. If we have $\{ab, c\} = a$, we have $ac\cdot b$; and if we have $ac\cdot b$, we have $\{ab, c\} = a$. For, if $\{ab, c\} = a$, we have $ca\cdot b$, by the definition

of $\{ab, c\}$. And if we have $ac{\cdot}b$, since we also have $\begin{pmatrix} ac{\cdot}a \\ ab{\cdot}a \end{pmatrix}$ (Law IV.), we have $\{ab, c\} = a$.

25. If $\{ab, c\} = c$, then $a = b = c$. For, if $\{ab, c\} = c$, we have $\begin{pmatrix} cc{\cdot}a \\ cc{\cdot}b \end{pmatrix}$, and thus $c = a$ and $c = b$ (Law III.).

26. Corresponding theorems to those given in §§16–19 could be proved with regard to unsymmetrical resultants, and also with regard to expressions such as $[ab\{cde\}]$ involving both symmetrical and unsymmetrical resultants. It is not, however, necessary to consider these, as it will presently be shown that unsymmetrical resultants can be expressed as symmetrical ones, and thus we need only consider formulæ relating to the latter. But, in order to arrive at this result, it is necessary to prove the formula of the next section.

27. We have $$\{\{ab, z\} c, z\} = \{\{ac, z\} b, z\}.$$

For let $\{ab, z\} = p$, $\{ac, z\} = q$, $\{pc, z\} = m$, $\{qb, z\} = n$.

Then we have $\begin{pmatrix} ab{\cdot}p \\ pz{\cdot}a \\ pz{\cdot}b \end{pmatrix}$, $\begin{pmatrix} ac{\cdot}q \\ qz{\cdot}a \\ qz{\cdot}c \end{pmatrix}$, $\begin{pmatrix} pc{\cdot}m \\ mz{\cdot}p \\ mz{\cdot}c \end{pmatrix}$, $\begin{pmatrix} qb{\cdot}n \\ nz{\cdot}q \\ nz{\cdot}b \end{pmatrix}$;

and I have to prove that $m = n$.

Since we have $\begin{pmatrix} pz{\cdot}b \\ mz{\cdot}p \end{pmatrix}$, we also have $mz{\cdot}b$ and $mb{\cdot}p$ (§ 8).

" " $\begin{pmatrix} pz{\cdot}a \\ mz{\cdot}p \end{pmatrix}$, " " $mz{\cdot}a$ (§ 8).

" " $\begin{pmatrix} zq{\cdot}a \\ zm{\cdot}c \\ ac{\cdot}q \end{pmatrix}$, " " $ma{\cdot}q$ (§ 10).

" " $\begin{pmatrix} mz{\cdot}a \\ ma{\cdot}q \end{pmatrix}$, " " $mz{\cdot}q$ (§ 8).

" " $\begin{pmatrix} ba{\cdot}p \\ bm{\cdot}p \\ ma{\cdot}q \end{pmatrix}$, " " $bq{\cdot}p$ (§ 11).

" " $\begin{pmatrix} zm{\cdot}p \\ zq{\cdot}c \\ pc{\cdot}m \end{pmatrix}$, " " $pq{\cdot}m$ (§ 10).

" " $\begin{pmatrix} bq{\cdot}p \\ pq{\cdot}m \end{pmatrix}$, " " $bq{\cdot}m$ (§ 8).

Thus, since we have $\begin{pmatrix} qb\cdot x \\ xs\cdot q \\ xs\cdot b \end{pmatrix}$ and $\begin{pmatrix} qb\cdot w \\ ws\cdot q \\ ws\cdot b \end{pmatrix}$, we have $w = x$ (§ 20).

28. It follows immediately that we have

$$\{\{ab, z\}\, c, z\} = \{\{ac, z\}\, b, z\} = \{\{bc, z\}\, a, z\},$$

and thus the entity represented by each expression is symmetrically related to a, b, and c. We may therefore represent it by the symbol $\{abc, z\}$, where the order of the letters a, b, c is immaterial. Furthermore, we may write

$$\{\{abc, z\}\, d, z\} = \{abcd, z\},$$

where the order of the four letters a, b, c, d is immaterial; and generally the expression $\{abcd \ldots pqrst, z\}$ will represent an entity of the system which is symmetrically related to a, b, c, d, ... p, q, r, s, t; and we have

$$\{\{abcd \ldots pqrs, z\}\, t, z\} = \{abcd \ldots pqrst, z\}.$$

Obverses.

29. If the collection of entities a, b, c, d, ... p, q, r, s, t comprises *all* the entities of the system, then $\{abcd \ldots pqrst, z\}$ will be an entity of the system which I shall denote by the symbol z'.

30. We have $zz'\cdot a$ whatever entity of the system a may be. For, let P denote the collection consisting of all the entities of the system except a. Then we have

$$\{\{P, z\}\, a, z\} = z',$$

and therefore we have $zz'\cdot a$, by the definition of an unsymmetrical resultant.

31. If u be such that we have $zu\cdot a$ whatever entity of the system a may be, then $u = z'$. For we have $zz'\cdot u$ (§ 30); and, since we have $zu\cdot a$ whatever entity of the system a may be, we have $zu\cdot z'$. Thus, since we have $\begin{pmatrix} zz'\cdot u \\ zu\cdot z' \end{pmatrix}$, we have $u = z'$ (§ 7).

32. We have $(a')' = a$.

For we have $\begin{pmatrix} zz'\cdot a' \\ aa'\cdot z \\ aa'\cdot z' \end{pmatrix}$ (§ 30), and therefore $a' = \{zz', a\}$ (§ 21).

Also, we have $\begin{pmatrix} zz'\cdot a \\ aa'\cdot z \\ aa'\cdot z' \end{pmatrix}$ (§ 30), and therefore $a = \{zz', a'\}$.

Hence $(a')' = \{ss', a'\} = a$. I shall call a' an *obverse* of a. If a is an obverse of b, b is an obverse of a. For, if $a = b'$, then $a' = (b')' = b$.

33. We cannot regard an entity a of the system and its obverse a' as equivalent without regarding all the entities of the system as equivalent. For we have $aa'\cdot x$ whatever entity of the system x may be (§ 30). Hence, if $a = a'$, we have $a = a' = x$ (Law III.), and thus every entity of the system is equivalent to a. If we assume that all the entities of the system are not equivalent, as we must if it is to have any properties at all, we may say that obverses are such as cannot be regarded as equivalent.

34. If we have $ab\cdot c'$, since we also have $bb'\cdot a$ (§ 30),
we have $b'c'\cdot a$,
and therefore also $ac\cdot b'$,
and $a'b'\cdot c$,
and $bc\cdot a'$,
and $a'c'\cdot b$.

Expression of Unsymmetrical Resultants as Symmetrical Resultants.

35. If $x = \{ab, c\}$, we have $\begin{pmatrix} ab\cdot x \\ cx\cdot b \\ cx\cdot a \end{pmatrix}$ (§§ 20, 21), and therefore we have also $\begin{pmatrix} ab\cdot x \\ bc'\cdot x \\ ac'\cdot x \end{pmatrix}$ (§ 34), and therefore $x = [abc']$ (§§ 12, 13).

Hence $\{ab, c\} = [abc']$. It is unnecessary, therefore, to have any special symbol for unsymmetrical resultants, as they can be expressed as symmetrical ones.

Symmetrical Resultants and Obverses.

36. We have $[aba'] = \{ab, a\} = b$ (§ 23).

37. If $x = [abc]'$, we have $x' = [abc]$ (§ 32), and therefore we have $\begin{pmatrix} ab\cdot x' \\ bc\cdot x' \\ ca\cdot x' \end{pmatrix}$ (§§ 12, 13), and therefore also $\begin{pmatrix} a'b'\cdot x \\ b'c'\cdot x \\ c'a'\cdot x \end{pmatrix}$ (§ 34), and thus $x = [a'b'c']$. Hence $[abc]' = [a'b'c']$.

Expression of Symmetrical Resultants as Unsymmetrical Resultants.

38. We have $[abc] = \{ab, c'\}$ (§ 35), and thus $[abc]$ is represented as an unsymmetrical resultant $\{ab, c'\}$; but we may also represent it as an unsymmetrical resultant without the introduction of obverses.

For
$$\begin{aligned} [abc] &= \big[[aba'][abb']\, c\big] && (\S\ 36) \\ &= \big[ab\,[a'b'c]\big] && (\S\ 16) \\ &= \big[ab\,[abc']'\big] && (\S\ 37) \\ &= \{ab, \{ab, c\}\}, \end{aligned}$$

a symbol which has no reference to obverses. It will, however, in general be more convenient and simple to adopt the reverse process, and represent unsymmetrical resultants as symmetrical ones; so that we have to deal with symmetrical resultants only.

Algebraical Treatment.

39. Suppose now that we consider symmetrical resultants of triads of entities of the system which are such that one member of each triad is always the entity z, which I shall accordingly call "the *constant z*." We may here regard each resultant as a function of the *two* remaining members of the triad of which it is the resultant; *e.g.*, we may regard the resultant $[abz]$ as a function of the two entities a, b; and thus to every two entities of the system there will correspond a definite third entity, and we shall have what I have termed in the before-mentioned Memoir (sec. 292) a *primitive algebra*.

40. Let, then, the resultant $[abz]$ be written simply as ab, and be called the *product* of a and b. Here, since

$$[zab] = [zba] \quad (\S\ 13),$$

we have
$$ab = ba,$$

and the primitive algebra is *commutative*. Also, since

$$\big[[abz]\, cz\big] = \big[[acz]\, bz\big] = \big[[bcz]\, az\big] \quad (\S\ 18),$$

we have
$$(ab)\, c = (ac)\, b = (bc)\, a,$$

and the algebra is *associative*.

We have also $aa = [aaz] = a$ (§ 14),

and $az = [azz] = z$ (§ 14).

41. We may, if we please, use the symbol 0 in lieu of z, since z is such that $az = z$ whatever entity of the system a may be; but it must be borne in mind that the entity 0, though an exceptional one as regards the algebra under consideration, because it is the *constant* of such algebra, does not differ in any respect from the other entities of the system, each of which may be taken as the constant of a precisely similar primitive algebra.

42. Taking now z' as the constant in lieu of z, let us write

$$[abz'] = a+b,$$

and call it the *sum* of the entities a, b. This second primitive algebra is *commutative* and *associative* just as the first was, and we have also

$$a+a = a, \quad a+z' = z'.$$

43. We may use the symbol ∞ in lieu of z', since $a+z' = z'$ whatever entity of the system a may be; but the remarks made in § 41 are, of course, equally applicable here. It will also be shown, in § 46, that the symbol ∞ is open to objection, and that it is better to use $0'$.

44. Considering now the compound algebra obtained by combining the two primitive algebras, we have

$$a(b+c) = \big[az\,[bcz']\big] = \big[[azb][azc]\,z'\big] \quad (\S\,16)$$

$$= ab+ac;$$

so that here multiplication is *distributive* as regards addition, as in the case of the ordinary algebra of quantity. We also have

$$a+bc = \big[az'\,[bcz]\big] = \big[[az'b][az'c]\,z\big] \quad (\S\,16)$$

$$= (a+b)(a+c);$$

so that here addition is distributive as regards multiplication, which is, of course, not the case with the ordinary algebra of quantity.

45. We have $a+z = [azz'] = a$ (§ 36),

and $az' = [azz'] = a$ (§ 36).

46. The identity $a+z=a$ is consistent with the employment of 0 in lieu of z; but the employment of ∞ in lieu of z' is not consistent with the identity $az'=a$. It would be consistent with $az'=a$ to employ the symbol 1 in lieu of z'; but this would not be consistent with the identity $a+z'=z'$. It would seem better, therefore, if we employ 0 in lieu of z, to employ the unexceptionable symbol 0′ in lieu of z', rather than ∞ or 1.

47. We have $ab+a=\big[[abz]\,az'\big]=\big[[aaz'][zaz']\,b\big]$ (§ 16)

$=[aab]$ (§§ 14, 36)

$=a$ (§ 14).

Similarly, we have

$$(a+b)\,a=aa+ab=a+ab=a.$$

48. We have $(ab)'=[abz]'=[a'b'z']$, (§ 37), $=a'+b'$;

and $(a+b)'=[abz']'=[a'b'z]$, (§ 37), $=a'b'$;

and generally $(abcd\ldots)'=a'+b'+c'+d'+\ldots$;

and $(a+b+c+d\ldots)'=a'b'c'd'\ldots.$

49. We have $a+a'=[aa'z']=z'$ (§ 36),

and $aa'=[aa'z]=z$,,

Observe here that, if we have

$$a+x=z'$$

and

$$ax=z,$$

then

$$x=a'.$$

For, since

$$x+a=z',$$

we have

$$(x+a)'=z \quad (\S\,32),$$

i.e.,

$$x'a'=z \quad (\S\,48).$$

Now $x=xz'$ (§ 45) $=x\,(a+a')=xa+xa'=z+xa'$

$=x'a'+xa'$

$=(x'+x)\,a'$

$=z'a'$

$=a'$ (§ 45).

Algebraical Expression of Resultants.

50. $[abc] = [ab\,[z'zc]]$ (§ 36) $= [[abz'][abz]\,c]$ (§ 16)

$= \Big[\big[ab\,[zz'z']\big]\,\big[zz'\,[abz]\big]\,c\Big]$ (§ 36)

$= \Big[\big[[abz][abz']\,z'\big]\,\big[zz'[abz]\big]\,c\Big]$ (§ 16)

$= \Big[\big[[abz']\,cz\big]\,[abz]\,z'\Big]$,,

$= (a+b)\,c+ab$

$= ac+bc+ab$

$= (a+b)(b+c)(c+a).$

Observe here the perfect symmetry of the result as regards the two primitive algebras of addition and multiplication. Also observe that the expressions $ac+bc+ab$ and $(a+c)(b+c)(a+b)$, though expressed by means of the constants z, z', are each equal to $[abc]$, and are therefore entirely independent of z and z', being functions of a, b, and c only.

51. Since $\{ab, c\} = [abc']$ (§ 35),

we have $\{ab, c\} = ab+ac'+bc'.$

52. We have seen (§ 25) that, if $\{ab, c\} = c$, then

$$a = b = c.$$

Thus, if $ab+az'+bz' = z,$

i.e., if $ab+a+b = z$ (§ 45),

i.e., if $a+b = z$ (§ 47),

then $a = b = z.$

Hence, generally, if the sum of any number of entities of the system $= z$, each $= z$. Similarly, it may be shown that, if the product of any number of entities of the system $= z'$, each of them $= z'$.

To express algebraically that we have ab·c.

53. The condition that we have $ab \cdot c$ is

$$[abc] = c \quad (\S\ 15).$$

If $[abc] = c$, then $[abc]\,c' = cc' = z$ (§ 49),

i.e., $(ab+bc+ca)\,c' = z$,

i.e., $abc'+bcc'+cc'a = z$,

i.e., $abc'+bz+az = z$ (§ 49),

i.e., $abc'+z+z = z$ (§ 40),

i.e., $abc' \quad = z$ (§ 45).

But, if we have $[abc] = c$,

we also have $[abc]' = c'$;

i.e., we have $[a'b'c'] = c'$ (§ 37).

Whence we can prove, in the same way as we proved that $abc' = z$, that $a'b'c = z$.

Thus, if we have $ab \cdot c$, we have

$$abc' = z \text{ and } a'b'c = z,$$

Conversely, if we have $abc' = z$ and $a'b'c = z$,

we have $ab \cdot c$. For

$$\begin{aligned}
[abc] = ab+bc+ca &= ab\,(c+c')+bc\,(a+a')+ca\,(b+b') && (\S\S\ 49, 45)\\
&= abc+abc+abc+abc'+ab'c+a'bc\\
&= abc+abc'+ab'c+a'bc && (\S\ 42)\\
&= c\,(ab+ab'+a'b)+abc'.
\end{aligned}$$

$$\begin{aligned}
\text{Thus } [abc]+a'b'c &= c\,(ab+ab'+a'b+a'b')+abc'\\
&= c\,(a+a')(b+b')+abc'\\
&= c+abc', && (\S\S\ 49, 45)\\
\textit{i.e.},\ [abc]+z &= c+z, && (\S\ 45)\\
\textit{i.e.},\ [abc] &= c,
\end{aligned}$$

i.e., we have $ab \cdot c$ (§ 15).

Full Sets.

54. If we take a finite number of entities $a, b, c, \ldots z$ of the system, and add to these the unsymmetrical resultants of each triad of them, and then add to the enlarged collection all their unsymmetrical

resultants, and so on, we obtain from the original finite number of entities an enlarged collection which may be said to be "derived" from the original collection. Since

$$[abc] = \{ab, \{ab, c\}\}, \quad (\S\, 38),$$

the derived collection will also contain all the symmetrical resultants as well as the unsymmetrical ones. The derived collection will not contain an infinite number of entities no two of which are equivalent; for, after a certain number of resultants are obtained, all that follow will be equivalent to one or other of those which have already been obtained. That this is so, will be obvious when it is remembered that every resultant may be expressed algebraically, and, since $aa = a$ and $a + a = a$ (§§ 40, 42), the number of non-equivalent algebraical expressions which can be formed with the finite number of letters $a, b, c, \ldots z$, is necessarily finite also. Thus to every finite collection of entities of the system there corresponds a collection consisting of a finite number of sets of equivalent entities, which collection is such that every resultant of three of its component entities is also a member of the collection. Such a collection I call a *full set.*

55. If z be any entity of a full set, the entity $\{abc \ldots p, z\}$ (§ 28), where the collection $a, b, c, \ldots p$ comprises all the entities of the full set, may be called *an obverse of z in the full set,* and be denoted by the symbol $\bar{z}$. The entity $\bar{z}$ will not, in general, be a true obverse z', though it will be so if any one of the entities $a, b, c, \ldots p$ is a true obverse of any other.

The entity $\bar{a}$ occupies, as regards the entities of the full set, exactly the same position as the entity a' occupies as regards the whole system. Thus, to take one property only, we have $a\bar{a} \cdot x$, whatever entity of the full set x may be. The proof is that of § 30, substituting "full set" for "system."

56. Where $\bar{a}$ is not equivalent to a', the true obverses of the various members of a full set compose another full set precisely similar to the former; and the two full sets, when combined together, also constitute a full set. As a simple example, let us start with the three entities a, b, z. From these we obtain the full set of eight entities,

$$\begin{array}{cccc} a, & b, & z, & ab, \\ a'b, & b'a, & a+b, & a'b+b'a, \end{array}$$

where the *obverse in the full set* of each of the members of the full set is given above or below it. Taking the true obverses of these, we

get the full set,

$$a', \quad b', \quad z', \quad a'+b'.$$
$$a+b', \quad b+a', \quad a'b', \quad ab+a'b',$$

where, as in the former case, the obverse in the full set of each of the members is given either above or below it. These two full sets, when combined, compose a fresh full set of sixteen members. Each of these is, however, of course, accompanied by an infinite number of equivalent entities.

57. To every pair of entities of a full set, there correspond certain entities of the full set, each of which is the odd member of a linear triad of which the pair of entities are the even members. In the case of certain pairs, each of these corresponding odd members is equivalent to one or other of these even members. In such case the pair may be said to be *adjacent* entities of the full set. The division of the pairs of a full set into adjacent and non-adjacent pairs completely defines the "form" of a full set. It is in this way that the form of a full set is defined in the before-mentioned Memoir (secs. 380–382).

58. The number of sets of equivalent entities in a full set is of the form 2^n (see the Memoir, sec. 363).

Linear Triads of which z and z′ are members.

59. When we treat the entity z as a constant, and represent the resultant $[zab]$ by the symbol ab in which z is suppressed, it is natural to take the same course in the case of the symbols representing linear triads of which z is a member. Thus we may represent the linear triad $az\cdot b$ by the symbol $a>b$, and the linear triad $ab\cdot z$ by the symbol $a*b$. Here $a*b$ and $b*a$ will represent the same triad, but $a>b$ and $b>a$ will not do so unless $a=b$.

60. If we have $az\cdot b$ and $bz\cdot c$, it is shown in § 8 that we have also $az\cdot c$ and $ac\cdot b$. Hence, if we have $a>b$ and $b>c$, we have $a>c$; so that the relation expressed by the symbol $>$ is *transitive*. We have also $ac\cdot b$, a triad not expressible in the same way as the others, because it does not involve z.

61. If we have $az\cdot b$ and $ac\cdot z$, we have also $bc\cdot z$ and $ac\cdot b$ (§ 8); thus, if we have $a>b$ and $a*c$, we have $b*c$ and $ac\cdot b$.

62. The algebraical condition that we have $ab\cdot z$ is (§ 53) $abz'=z$,

and $a'b'z = z$. The second of these equations is an identity (§ 40), and the first may be written $ab = z$ (§ 45), which is therefore the algebraical expression of the fact that we have $ab \cdot z$. The equation $a' + b' = z'$ expresses the same thing.

63. It follows immediately, from §§ 34, 59, and 62, that each of the twelve different symbols in each of the four following columns indicates the existence of the same relation between the three entities a, b, z; so that four different relations only between a, b, z are represented by the forty-eight different symbols—

$a * b$	$a' * b$	$a * b'$	$a' * b'$
$b * a$	$b * a'$	$b' * a$	$b' * a'$
—	—	—	—
$a' > b$	$a > b$	$a' > b'$	$a > b'$
$b' > a$	$b' > a'$	$b > a$	$b > a'$
—	—	—	—
$ab \cdot z$	$a'b \cdot z$	$ab' \cdot z$	$a'b' \cdot z$
$a'b' \cdot z'$	$ab' \cdot z'$	$a'b \cdot z'$	$ab \cdot z'$
$b'z \cdot a$	$bz' \cdot a$	$bz \cdot a$	$b'z' \cdot a$
$bz' \cdot a'$	$b'z \cdot a'$	$b'z' \cdot a'$	$bz \cdot a'$
$a'z \cdot b$	$az \cdot b$	$az' \cdot b$	$a'z' \cdot b$
$az' \cdot b'$	$a'z' \cdot b'$	$a'z \cdot b'$	$az \cdot b'$
—	—	—	—
$ab = z$	$a'b = z$	$ab' = z$	$a'b' = z$
$a' + b' = z'$	$a + b' = z'$	$a' + b = z'$	$a + b = z'$.

64. It is shown in § 53 that the existence of the linear triad $ab \cdot c$ may be expressed by the equations

$$abc' = z \quad \text{and} \quad a'b'c = z.$$

The second of these may be written

$$(a + b)'c = z \ (\S\, 48),$$

and thus we may express the fact that a, b, c compose the linear triad $ab \cdot c$ by the symbols

$$c > ab,$$

$$a + b > c.$$

65. There is no object in employing any fresh symbols to represent linear triads in which z' is the constant, as it appears from § 63 that such triads are represented by the symbols given for the cases in which z is the constant. Thus $az'{\cdot}b$ is represented by $b>a$, and $ab{\cdot}z'$ by $a' * b'$.

66. The following relations hold, whatever entity of the system a may be:—

$$a>z,$$

$$z'>a.$$

Algebraical Proof of Laws I.—IV.

67. In the preceding sections I have deduced the algebraical treatment of the base system from the laws of §§ 5, 6. I now propose, taking a different course, to assume the validity of the algebraical treatment and Law V., and thence to prove the truth of Laws I., II., III., and IV.

68. Here the three entities a, b, c will be said to compose a linear triad $ab{\cdot}c$, if we have

$$abc'=z \quad \text{and} \quad a'b'c=z \ (\S\ 53).$$

69. To prove Law I., I have to show that, if the four equations

$$apb'=z, \quad a'p'b=z, \quad cpd'=z, \quad c'p'd=z$$

hold, then q exists such that we also have

$$adq'=z, \quad a'd'q=z, \quad bcq'=z, \quad b'c'q=z.$$

We have $ab'c'd=ab'c'd\,(p+p')=ab'c'dp+ab'c'dp'$

$$=(ab'p)\,c'd+(c'dp')\,ab'$$

$$=z,$$

since $ab'p=z$ and $c'dp'=z$.

We also have $a'bcd'=a'bcd'p+a'bcd'p'$

$$=(cd'p)\,a'b+(a'bp')\,cd'$$

$$=z,$$

since $cd'p=z$ and $a'bp'=z$.

Consider now the entity q, which is such that

$$q = (ad+bc)\,x+(a+d)(b+c)\,x',$$

where x is any entity of the system. This entity q is a possible member of the system, and therefore, by Law V., it is a member.

We have $q' = (a'+d')(b'+c')\,x+(a'd'+b'c')\,x'.$

Hence
$$qa'd' = a'd'bcx = z, \quad \text{since } a'd'bc = z,$$
$$q'ad = adb'c'x' = z, \quad \text{,, } adb'c' = z,$$
$$qb'c' = adb'c'x = z, \quad \text{,, } adb'c' = z,$$
$$qbc = a'd'bcx = z, \quad \text{,, } a'd'bc = z.$$

The law is therefore proved.

70. To prove Law II., I have to show that, if the four equations

$$abp' = z, \quad a'b'p = z, \quad cpd' = z, \quad c'p'd = z$$

hold, then q exists such that we have

$$aqd' = z, \quad a'q'd = z, \quad bcq' = z, \quad b'c'q = z.$$

The method of proof is precisely similar to that of the preceding section.

71. To prove Law III., I have to show that, if we have

$$abc' = z, \text{ and } a'b'c = z, \text{ and } a = b,$$

then
$$a = b = c.$$

Since $a = b$,
$$ac' = aac' = abc' = z,$$
$$a'c = a'a'c = a'b'c = z.$$

Hence
$$a = ac+ac' = ac+z = ac+a'c = c.$$

72. To prove Law IV., I have to show that, if we have $a = b$, we also have

$$ab'c = z, \quad ab'c' = z, \quad a'bc = z, \quad a'bc' = z.$$

Since $a = b$, $ab' = bb' = z$ and $a'b = a'a = z.$

Thus
$$ab'c = zc = z,$$
$$ab'c' = zc' = z,$$
$$a'bc = zc = z,$$
$$a'bc' = zc' = z.$$

The four laws are therefore all proved.

Representation of Non-linear Triads.

73. Hitherto nothing has been said as to the representation of non-linear triads. No special symbol for these has been required. It may, however, on occasion be desirable to denote the fact that a, b, c is not a linear triad $ab\cdot c$. We may do this by the symbol $\overline{ab\cdot c}$. Observe, however, that this symbol does not denote the fact that a, b, c do not form a linear triad, but only that they do not form a linear triad $ab\cdot c$; they may, in fact, form the linear triad $bc\cdot a$ or $ac\cdot b$. To denote the fact that a, b, c is not a linear triad at all, we must, in this method of representation, have $\overline{ab\cdot c}$, $\overline{ac\cdot b}$, and $\overline{bc\cdot a}$. We may, however, conveniently employ the symbol $a\cdot b\cdot c$ to indicate merely that a, b, c compose a linear triad without specifying which of the three entities is the odd member, and then $\overline{a\cdot b\cdot c}$ will denote the fact that a, b, c do not compose a linear triad.

74. We may, in the same way, employ the symbols $\overline{a * b}$ or $\overline{a' > b}$ to denote the same thing as $\overline{ab\cdot z}$. The non-existence of the linear triad $ab\cdot z$ will also be indicated by each of the inequalities

$$ab \neq z \quad \text{and} \quad a' + b' \neq z'.$$

75. Since, if we have $ab\cdot c$, we have

$$abc' = z \quad \text{and} \quad a'b'c = z \ (\S\ 53),$$

if we have $\overline{ab\cdot c}$, then either

$$abc' \neq z \quad \text{or} \quad a'b'c \neq z.$$

Obverse Triads, Tetrads, &c.

76. We have seen (§ 34) that, if we have

$$ab\cdot c',$$

we have also $a'b'\cdot c$,

and

and $a'c'\cdot b$,

and $bc\cdot a'$,

and $b'c'\cdot a$.

The relation between a, b, c which is expressed by the fact that we have $ab\cdot c'$ is therefore a symmetrical one. I shall represent the triad

a, b, c between which this relation exists by the symbol $\cdot abc\cdot$, and shall term the triad an *obverse* triad, for the following reason :—

77. If we regard two of the three entities of the triad $\cdot abc\cdot$, say a and b, as equivalent, then, since we have $ab\cdot c'$, it follows from Law III. that we have $a = b = c'$, and thus we cannot regard all the three entities a, b, c as equivalent (§ 33). The triad is thus analogous to an obverse pair a, a', and I accordingly call it an obverse triad.

78. The condition that a, b, c may be an obverse triad $\cdot abc\cdot$ is that we have $ab\cdot c'$ (§ 76), that is, that we have

$$abc = z, \quad \text{and} \quad a'b'c' = z \ (\S\ 53).$$

79. If we have $\quad abcd = z, \quad \text{and} \quad a'b'c'd' = z,$

we may call a, b, c, d an *obverse* tetrad, and represent it by $\cdot abcd\cdot$; for observe that, if any two of the entities of the tetrad, say a, b, are equivalent, we have

$$acd = z, \quad \text{and} \quad a'c'd' = z;$$

so that a, c, d is an obverse triad, and therefore all the four entities of the triad cannot be regarded as equivalent.

80. Generally, if we have

$$abcd \ldots p = z \quad \text{and} \quad a'b'c'd' \ldots p' = z,$$

we may call a, b, c, $d \ldots p$ an *obverse* collection, and represent it by $\cdot abcd \ldots p\cdot$.

81. It is obvious, from the preceding section, that, if we have $\cdot abcd \ldots p\cdot$, we have also $\cdot a'b'c'd' \ldots p'\cdot$.

82. Obverse collections are distributed through the system in accordance with the following laws, in which P represents a collection of any number of entities of the system :—

Law A.—If we have $\cdot P\cdot$, we also have $\cdot Pa\cdot$.

Law B.—If we have $\cdot Pa\cdot$ and $\cdot Pa'\cdot$, we also have $\cdot P\cdot$.

Flat Collections.

83. A collection of entities of the system a, b, c, d, p, q, r, s, which is such that we have $\cdot abcdp'q'r's'\cdot$, and therefore also $\cdot a'b'c'd'pqrs\cdot$ (§ 81), may be represented by the symbol $abcd\cdot pqrs$ or $pqrs\cdot abcd$, which is precisely analogous to $ab\cdot c$, as the latter represents a triad a, b, c which is such that we have $\cdot abc'\cdot$ and $\cdot a'b'c\cdot$.

84. The collection *abcd·pqrs* is such that, if a, b, c, d are all regarded as equivalent, and p, q, r, s are also all regarded as equivalent, then the whole collection must be regarded as equivalent.

85. I propose, for reasons which will appear later (§ 150), to call the collection *abcd·pqrs* a *flat* collection.

86. If a, b, c, d is a flat collection, it may be any one of the collections

$$a\cdot bcd$$
$$b\cdot cda$$
$$c\cdot dab$$
$$d\cdot abc$$
$$ab\cdot cd$$
$$ac\cdot db$$
$$ad\cdot bc.$$

If we merely wish to denote the fact that it is a flat collection, without specifying which of the foregoing forms it takes, we may, as in the case of a linear triad (§ 73), represent it by

$$a\cdot b\cdot c\cdot d,$$

and similarly in the case of flat collections of any number of entities.

87. If we have *abc·def*, we have $\cdot abcd'e'f'\cdot$ (§ 83), and therefore we have $\cdot abcpd'e'f'\cdot$ and $\cdot abcd'e'f'p'\cdot$ (§ 82), and therefore also *abcp·def* and *abc·defp*, whatever entity of the system p may be.

88. Thus, if we have $a\cdot b\cdot c\cdot d\cdot e\cdot f$, we have $a\cdot b\cdot c\cdot d\cdot e\cdot f\cdot p$, whatever entity of the system p may be.

89. Observe that, if we have $a\cdot b$, then $a=b$; and if $\cdot ab\cdot$, then $a=b'$.

Restricted Collections.

90. Every collection of entities of the system is either a *flat* collection or an *obverse* collection, or neither. In the latter case I shall say that it is *unrestricted;* so that a *restricted* collection is one which is either a flat or an obverse collection.

91. I shall denote a restricted collection a, b, c, d by the symbol

$$\cdot a\cdot b\cdot c\cdot d\cdot.$$

92. If we have $\cdot a\cdot b\cdot$, then either $a = b$, or $a = b'$ (§ 89).

93. If we have $\cdot a \cdot b \cdot c \cdot$, then we have either $ab \cdot c$,

or	$ac \cdot b$,	*i.e.*, $a'b \cdot c$	(§ 34),
or	$bc \cdot a$,	*i.e.*, $ab' \cdot c$	(§ 34),
or	$\cdot abc \cdot$,	*i.e.*, $a'b' \cdot c$	(§ 76).

94. It follows immediately, from §§ 82, 87, that, if we have

$$\cdot a \cdot b \cdot c \cdot d \cdot e \cdot f \cdot,$$

we have $$\cdot ab \cdot c \cdot d \cdot e \cdot f \cdot p \cdot,$$

whatever entity of the system p may be.

95. It follows, from §§ 81, 83, that, if a collection is restricted, so is every collection which can be formed by taking, in lieu of some or all of the members of the collection, the obverses of those members. Thus, if we have

$$\cdot a \cdot b \cdot c \cdot d \cdot,$$

we have $$\cdot a' \cdot b \cdot c' \cdot d \cdot,$$

and $$\cdot a' \cdot b' \cdot c' \cdot d \cdot.$$

Unrestricted Collections.

96. We may employ lines, as in §§ 73, 74, to indicate that collections are not obverse, or flat, or restricted. Thus,

$$\overline{\cdot abcd \cdot}, \quad \overline{ab \cdot cd}, \quad \overline{\cdot a \cdot b \cdot c \cdot d \cdot}.$$

97. I have now completed the discussion of the general properties of the base system, and would draw particular attention to the fact that those properties have been deduced by a consideration of the "form" of the system only, *i.e.*, of the mode in which the linear triads are distributed through the system. I pass on to show that this system is precisely that which is under consideration by the logician, when discussing the relations of classes to each other.

The Logical Theory of Classes.

The Law of Continuity.

98. In any particular investigation of class relations, the logician deals, of course, with a finite number of classes only; but in the logical theory such classes are conceived of as a selection from an infinite number of possible classes, forming a system which contains classes that bear every possible class relation to the selected ones.

The same thing occurs in the case of geometrical points, for the geometrician regards any finite set of points as forming part of the infinite system of points which make up geometrical space. Thus Law V. of § 6 applies to the system of entities which we call *classes*.

The Algebraical Treatment of Classes.

99. Let A, B, C, ... denote classes, and let A', B', C', ... denote the complementary classes, viz.: let A' contain nothing which A does, and everything which A does not. Let Z be that class which contains nothing, and therefore let Z' be the class which contains everything.

Let AB, say the *product* of A and B, denote that class which contains everything that is contained in both A and B, and nothing else.

Let $A+B$, say the *sum* of A and B, denote that class which contains everything that is contained in either A or B or both, and nothing else.

Then we have

$$AB = BA, \qquad A+B = B+A,$$

$$(AB)C = (AC)B = (BC)A, \quad (A+B)+C = (A+C)+B = (B+C)+A,$$

$$AA = A, \qquad A+A = A,$$

$$AA' = Z, \qquad A+A' = Z',$$

$$AZ = Z, \qquad A+Z' = Z',$$

$$AZ' = A, \qquad A+Z = A,$$

$$A(B+C) = AB+AC,$$

$$A+BC = (A+B)(A+C),$$

$$(A+B)' = A'B',$$

$$(AB)' = A'+B'.$$

We have, therefore, precisely the algebra with which we have been dealing in the case of the base system (§§ 39–49).

The "Form" of a System of Classes.

100. It accordingly follows, from §§ 67–72, that, if A, B, C, a triad of classes such that

$$ABC' = Z$$

$$A'B'C = Z,$$

be represented by the symbol $AB \cdot C$, and be termed the *linear triad*

$AB \cdot C$, then linear triads of classes are distributed through the whole system of classes in accordance with the Laws I.–IV., given in § 5.

101. Moreover, the algebra is competent to represent *all* the logical relations which exist between classes, and thus no other laws need be added to Laws I.–V. All, therefore, that is essential in the logical system of classes, so far as the logical theory is concerned, is that it is of precisely the same "form" as the base system.

Linear Triads involving the Non-existent and Universal Classes.

102. Regarded as members of such a base system, the classes Z and Z' are undistinguished in their relations and properties from any other classes. The reason why they appear to hold exceptional positions is merely that they are the two entities of the system which are taken as *constants*.

103. This being so, it is, as is pointed out in § 59, natural in the representation of relations involving those constants to refrain from explicitly referring to them. Accordingly, just as in § 59 the existence of the triad $ax \cdot b$ is represented by the symbol $a > b$, so we find the fundamental relation between the three classes A, Z, B, which is expressed by saying that they compose a linear triad $AZ \cdot B$, is usually expressed by the statement "A contains B."

Linear Triads of Classes in general.

104. If we adopt this mode of expressing the existence of the relation $AZ \cdot B$, we see, by § 64, that we may express the existence of the relation $AB \cdot C$ by the statements

"$A + B$ contains C,"

"C contains AB,"

or, in other words, C contains all that is contained in both A and B, and nothing except what is contained in either A or B.

105. We may, therefore, graphically represent the existence of the fundamental linear relation which is expressed by $AB \cdot C$, as in Fig. 1,

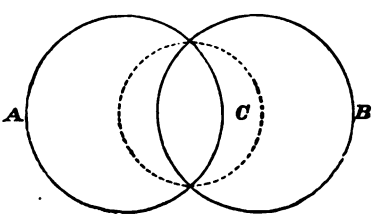

Fig. 1.

where the areas enclosed by the continuous circles represent the classes A and B, and that enclosed by the dotted circle the class C.

Relations between two Classes.

106. It will readily be seen from the figure that, if we take Z, *i.e.*, the class which contains nothing, in place of B, so that we have $AZ \cdot C$ or $A > C$, the class A will contain the class C.

107. It will also be seen that, if we take Z in place of C, so that we have $AB \cdot Z$ or $A * B$, the classes A and B will have no common part.

108. It will now, I think, be clear that the relations of inclusion $A > B$, and mutual exclusion $A * B$, which are usually regarded as relations between *two* classes A, B, are really relations between three classes, viz., A, B, Z.

109. I may here refer to the relations dealt with in §§ 60, 61. In the case of classes, § 60 shows that, if A contains B, and B contains C, then A contains C. But it also shows that we have $AC \cdot B$, a conclusion which is usually omitted, because the fundamental character of the linear triad has not been recognised.

110. So § 61 shows that, if A contains B, but excludes C, then B excludes C, and we also have $AC \cdot B$.

The Symmetrical Resultant of Three Classes.

111. The symmetrical resultant $[ABC]$ of three classes A, B, C, which are represented by the areas enclosed by the three circles in

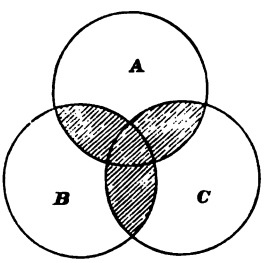

Fig. 2.

Fig. 2, will be represented by the shaded portion of the area, which is

$$AB+BC+CA = (A+B)(B+C)(C+A) \quad (\S\ 50).$$

112. It will readily be seen that, if we take the class Z, in place of C, the symmetrical resultant

$$[ABZ] \equiv AB$$

is the class which contains that which both A and B contain, and nothing else; which accords with § 99.

113. It will also be seen that, if we take the class of Z' in place of C, then

$$[ABZ'] \equiv A+B$$

is the class which contains that which either A or B or both contain, and nothing else; which accords with § 99.

114. It is unnecessary to go through the various sections in which I have dealt with the properties of the base system, and to point out the particular dress which their contents will assume when the entities are classes. It is sufficient for my present purpose to have established that all that is essential in classes, as dealt with in the logical theory, is that they are entities composing a system of which the "form" is that defined in §§ 2–6.

The Logical Theory of Statements.

The "Form" of a System of Statements.

115. It will be interesting here briefly to point out that *statements* as dealt with in the logical theory also compose a system of precisely the same form as the base system.

Truisms and Falsisms.

116. Here our constants z and z' are a *falsism* and a *truism* respectively. Logically speaking, all truisms are equivalent, as also all falsisms.

117. It is to be observed that to regard a statement as *true* is merely to regard it as a truism, and to regard it as *false* is to regard it as a falsism.

Contradictory Statements.

118. Obverse statements a and a' are such as are called *contradictory*. Since a and a' cannot be regarded as equivalent (§ 33), it is obvious that they cannot both be regarded as equivalent to a truism z', or a

falsism z. That is (§ 117), they cannot both be regarded as true, or both as false.

Implication and Inconsistency.

119. If we have $az \cdot b$, *i.e.*, $a > b$, the statement b is said to *imply* the statement a. Here, if $a = z$, *i.e.*, if a be false, it follows, from Law III., that b must also be false.

120. If we have $ab \cdot z$, *i.e.*, $a * b$, the statements a and b are said to be *inconsistent*. They can only be regarded as equivalent by being also regarded as equivalent to z, *i.e.*, by being both regarded as false.

"And" and "Or."

121. The product ab of the two statements a, b is the statement "a and b."

122. The sum $a+b$ of the two statements a, b is the statement "a or b," where, however, the "or" would more properly be written $\frac{and}{or}$ as in mercantile documents, so that we may replace the statement "a or b" by the equivalent statement "either a or b or both."

Linear Triads of Statements.

123. The existence of the linear triad $ab \cdot c$ may be expressed by the symbols $a+b > c$ and $c > ab$ (§ 64).

If, therefore, the statements a, b, c compose the linear triad $ab \cdot c$, the statement "c" implies the statement "a or b," and the statement "a and b" implies the statement "c." We know by Law III. that, if we regard a and b as equivalent, we must regard c as equivalent to them.

The Symmetrical Resultant of Three Statements.

124. The symmetrical resultant of the three statements a, b, c is the statement "a and b, or b and c, or c and a," which is equivalent to the statement "a or b, and b or c, and c or a."

Relations, &c., involving Two Statements.

125. It is important to note that the relations of implication $a > b$, and of inconsistency $a * b$, and the statements "a and b" and "a or b," each of which would usually be regarded as involving *two* statements only, have reference to the constant statement z, and thus really involve three statements.

The "Absolute" in Logic.

126. The *constants* in the logical theories of classes and statements occupy the same position as the "absolute" does in geometry; and when we treat relations such as those of "inclusion" and "exclusion" of "implication" and "inconsistency," as relations between *two* classes or *two* statements, and not as *linear triad* relations involving the constants as well as the two classes or statements, we are doing precisely the same thing as the geometrician does who regards "distance" as a relation between *two* points, and not as one involving the geometrical "absolute" as well as those two points.

The Geometrical Theory of Points.

Linear Sets.

127. I turn now to the consideration of certain sets of entities which can be selected from those which compose the base system—*i.e.*, of certain *loci* of that system, as we may term them. We have seen in § 8 that, if we have $\begin{pmatrix} ab\cdot c \\ ad\cdot b \end{pmatrix}$, we have also $\begin{pmatrix} ad\cdot c \\ cd\cdot b \end{pmatrix}$; and we have also seen in § 7 that, if we have $\begin{pmatrix} ef\cdot g \\ eg\cdot f \end{pmatrix}$, then $f = g$.

It follows from these two results that, if four entities of the fundamental system are such that no two are equivalent, and that every triad of these is a linear triad, then the set of four entities must be of one of two forms, viz.: it must either be such as a, b, c, d, where we have

$$\begin{aligned} &ad\cdot b \\ &ad\cdot c \\ &cd\cdot b \\ &ab\cdot c; \end{aligned}$$

or such as p, q, r, s, where we have

$$\begin{aligned} &pq\cdot r \\ &pq\cdot s \\ &rs\cdot p \\ &rs\cdot q. \end{aligned}$$

128. Furthermore, it follows also that, if we have a set of five entities of the system, no two of which are equivalent, and such that every triad of the set is a linear triad, then no tetrad of the set can be of the second of the two forms, but all must be of the first.

129. Thus, in a set of more than four entities of the base system, no two of which are equivalent, and every three of which compose a linear triad, every tetrad of the entities of the set is of one definite form. I shall call such a set a *linear set.*

130. A linear set may contain any number of entities. If it be such that there is no entity of the system outside the set which makes a linear triad with each pair of entities of the set, and is not equivalent to any entity of the set, the set may be said to be *complete.*

131. If we conceive of the odd member of a linear triad as "lying between" the two even members, we may conceive of the entities of a linear set as forming a *line.* In such a line the four entities a, b, c, d, which are such that we have

$$ad\cdot b,$$
$$ad\cdot c$$
$$cd\cdot b$$
$$ab\cdot c$$

will lie in the order $acbd$.

132. If we add to a linear set all those entities which are equivalent to the various members of it, we may regard the sets of equivalent members as massed together at one point, and then, instead of a line of single entities, we shall have a line of sets of equivalent entities. I shall suppose that all these equivalent entities are included in a linear set, whether it be complete or not. The definitions of §§ 129, 130 must be modified accordingly.

133. A linear set cannot include an entity a and also its obverse a'; for, if b be any other entity of the set, a, a', b would be an obverse triad and not a linear one.

134. Two linear sets may have any number of non-equivalent entities in common. In the language of geometry, they may cut each other any number of times.

Geometric Sets.

135. We may, however, take a set, or *locus*, of entities of the base system such that no two linear sets of the entities comprised in the locus can have two non-equivalent entities in common.

136. Such a locus would be fully defined by the following law as to the distribution of linear and non-linear triads within it, where the notation is that of § 73 :—

Law VI.—If we have $a \cdot p \cdot q$

$b \cdot p \cdot q$

and $\overline{a \cdot b \cdot p}$;

then $p = q$.

137. This locus I shall call a *geometric set*. It is precisely the set of entities which is under consideration by the geometrician when he is considering the system of points which make up flat space of unlimited dimensions. Here three points which are collinear correspond to a linear triad, the point which lies between the other two corresponding to the odd member of the triad. I proceed to show that this is so—*i.e.*, that Laws I.–IV. of § 5, Law V. of § 6, and Law VI. of § 136 completely define the system of entities which we call points, so far as its geometrical properties are concerned. I will first show that all those laws are true in the case of triads of collinear points, and will afterwards point out that they are sufficient.

138. As already mentioned in § 98, in the geometrical theory of points, the geometrician regards points as composing an infinite system containing points bearing every possible geometrical relation to any selected collection. Thus Law V. of § 6 applies to the case of a system of points.

139. If the three points a, b, p of Fig. 3 are collinear, and also the

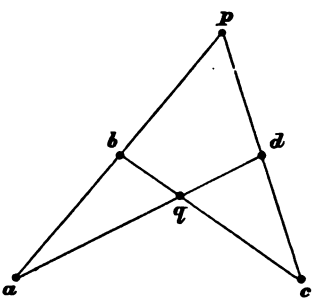

Fig. 3.

three points c, d, p, the point b lying between the points a and p, and the point d lying between the points c and p; then a point q exists such that a, q, d are a collinear triad, q lying between a and d, and that b, q, c are also a collinear triad, q lying between b and c. Thus Law I. of § 5 holds in the case of points.

140. So also in the case of Law II. of § 5. If the three points

a, p, b in Fig. 4 are collinear, p lying between a and b, and the three

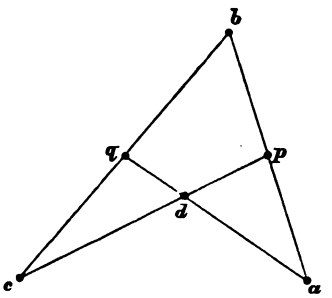

Fig. 4.

points c, d, p are also collinear, d lying between c and p; then a point q exists such that a, d, q are collinear, d lying between c and q, and such that c, q, b are collinear, q lying between c and b.

141. Again, if a, c, b are collinear, and c lies between a and b, and if a and b move together and coincide, *i.e.*, become equivalent, c which lies between them must also coincide with them, *i.e.*, become equivalent to them. Thus Law III. of § 5 also holds.

142. If two points a and b are coincident, *i.e.*, equivalent, then a, b, c is a collinear triad, whatever point c may be, and b may be regarded as lying between a and c, and a as lying between b and c. Thus Law IV. of § 5 also holds good.

143. Lastly, two straight lines cannot have more than one non-equivalent—*i.e.*, non-coincident—point in common. Thus, if two non-coincident lines have two points in common, they must be coincident points. Thus, if we have $\overline{a \cdot b \cdot p}$, *i.e.*, if the three points a, b, p are not collinear, the two straight lines which pass respectively through a, p and b, p can only have p, or points equivalent to p, in common. If, then, q lies on the straight line through a and p, *i.e.*, if we have $a \cdot p \cdot q$, and if q lies also on the straight line through b and p, *i.e.*, if we have $b \cdot p \cdot q$, we must have $p = q$, and so Law VI. of § 136 holds.

144. The laws are thus all true. Are they sufficient? In answering this question, we need not trouble ourselves about metrical geometry; that is now fully understood to be included in the more general descriptive geometry.

145. In the descriptive geometry of a system of points, we consider *their configurations*. Certain points are collinear, certain coplanar,

certain lie on a conic, and so on. These various configurations may be defined by means of algebraical equations between coordinates. It is an almost immediate deduction from this that such configurations may all be defined by statements as to certain triads of points being collinear.

146. Thus the condition that the four points a, b, c, d may be coplanar is that there exists a point p such that the triads a, b, p and c, d, p are collinear; the condition that the five points a, b, c, d, e may lie in flat space of three dimensions is that there exist points l and m such that the triads a, b, l and c, d, m and l, m, e are all collinear; and the condition that the six points a, b, c, d, e, f may lie on a conic is that the seven triads

$$a, b, l \quad b, c, m \quad c, d, n \quad l, m, n$$
$$d, e, l \quad e, f, m \quad f, a, n$$

are all collinear triads.

147. All, therefore, that is necessary in a system of points, in order that it may have those properties which are considered in descriptive geometry, is that the collinear triads of those points may be distributed through the system in the manner which is distinctive of such a system. This mode of distribution is completely defined by the laws which we have been considering, and they are accordingly sufficient to determine all the geometrical properties of the system.

148. Starting from these laws, there would be no difficulty in developing from them, without introducing any other consideration, these various geometrical properties; but this is all mere matter of detail which can readily be worked out by any one who has a knowledge of descriptive geometry, and its elaboration would be out of place here.

Algebraical Treatment of Geometric Sets.

149. I must, however, refer in the briefest possible way to one matter of importance, viz.—the way in which the algebraical treatment of the system arises. It will be sufficient to consider the case of a set of collinear points. It is usual, in such algebraical treatment, to consider certain entities termed "quantities," viz.,—with each point in the set of collinear points there is associated a "quantity," its distance from a fixed point O of the set. But it is quite unnecessary to introduce any entities into consideration except the points themselves; these will give rise to a complete algebra, of precisely the

same character as the algebra of quantity, which will answer every purpose which is served by the latter. We have already seen in §§ 39 *ff* how an algebra arises in the case of the complete base system, by means of which we can discuss its properties. The algebra of the set of entities of the system which corresponds to a set of collinear points, though different in character from the former algebra, is derived in a precisely similar way. We have here three constant points i, z, u of the collinear set, and then, if we take any two other points a, b of the set, we obtain two different resultant points of the five points a, b, i, z, u. The first of these resultant points I call the *sum* of the points a, b, and write $a+b$, and thus get a primitive algebra which is associative and commutative, and such that

$$a+i=i, \quad a+z=z.$$

The second resultant point I call the *product* of the points a, b, and write ab, and thus get a second primitive algebra which is also associative and commutative, and such that

$$ai=i, \quad az=z, \quad au=a.$$

The second algebra is distributive as regards the first, but the first is not distributive as regards the second. The compound algebra which we obtain by combining the two primitive algebras is thus precisely the same as the ordinary algebra of quantity, the points i, z, u, which may be any three points whatever of the collinear set, corresponding respectively to the quantities ∞, 0, and 1. This point algebra is considered more fully in my "Memoir on the Theory of Mathematical Form" (sec. 309), and the mode of deriving the resultant points is there indicated; so I have here merely referred to the matter in the barest outline.

Flat Collections.

150. I said, in § 85, that reasons would be given later why the collections of entities there referred to could appropriately be called "flat" collections. Those reasons may now be given. Every such collection of n entities, say every *flat n-ad*, is a collection which, as a part of a geometric set, consists of n points which lie in flat space of $n-2$ dimensions. Thus a flat or linear triad of a geometric set consists of three collinear points, *i.e.*, three points lying in space of one dimension; so a flat tetrad consists of four points lying in a plane, *i.e.*, in space of two dimensions, and so on. The different sorts of flat collections of n dimensions (see § 86) correspond to the different configurations which a flat collection can have. Thus, to take the case of

a flat tetrad $a{\cdot}b{\cdot}c{\cdot}d$: this can have two forms of which $a{\cdot}bcd$ and $ab{\cdot}cd$ are types. If we have $a{\cdot}bcd$, a lies within the triangle formed by b, c, and d; while, if we have $ab{\cdot}cd$, a and b lie on opposite sides of the straight line through c and d, and c and d lie on opposite sides of the straight line through a and b.

151. I have now completed what I have to say about the ordinary geometrical theory of points, and I hope that I have succeeded in making clear the nature of the relation which exists between a system of points and a system of classes.

Extended Geometric Sets.

152. I desire, however, to add one or two paragraphs with regard to an extension of the ordinary theory of geometric space which is suggested by the foregoing investigation. It follows from the definition of a geometric set (§§ 135–137) that no geometric set can contain the obverse of any point which is a member of the set. For, if points a and a' were both members of the geometric set, then, if b and c are any other members of the set, we have $aa'b$ and $aa'c$ (§ 30), and therefore, since a does not equal a', by Law VI. we have also $a{\cdot}b{\cdot}c$ and $a'{\cdot}b{\cdot}c$. Thus all the points of the geometric set must be collinear, in which case the set cannot include both a and a' (§ 133).

153. The obverses, then, of the various points which compose a geometric set will themselves compose a set distinct from the former. Also, since, if three points a, b, c compose a linear triad, their obverses a', b', c' will also compose a linear triad (§ 34), it follows that the set composed of the obverses will be a geometric set precisely similar to the original set. These two geometric sets may be said to be *conjugate geometric sets.*

154. Let us now consider the set obtained by combining two conjugate geometric sets. This will be a symmetrical set which I shall call an *extended geometric set.* It does not, as might for a moment be supposed, break up into two conjugate geometric sets (those out of which it was formed) in one way only, but it so breaks up in an infinite number of ways. The two conjugate geometric sets may, in fact, be regarded as analogous to two hemispheres, which, when put together, compose a complete sphere.

155. In an extended geometric set, in place of a complete linear set (§ 130), or straight line, which is such that every triad of its component points is a linear triad, we have an *extended linear set*, which is such

that every triad of its component points is a *restricted* triad (§ 90). The points in such an extended linear set may be regarded as forming a circuit. This circuit may be divided into two ordinary straight lines in an infinite number of ways.

156. In the same way, in place of the set of points in ordinary geometric space which compose a plane, *i.e.*, which are such that every tetrad of them is a *flat tetrad* (§ 150), in the extended geometric set we have an extended plane which is such that every tetrad of its component points is a *restricted tetrad.*

157. And generally, in place of the set of points in flat space of infinite dimensions which compose *a flat space of n dimensions*, *i.e.*, are such that every $(n+2)$-ad of them is a flat $(n+2)$-ad, we have in our extended geometric space a set of points composing a *restricted space of n dimensions*, *i.e.*, such that every $(n+2)$-ad of them is a restricted $(n+2)$-ad.

158. It would carry me beyond the purpose of this paper to develop further the consequences of thus regarding geometric space of any dimensions as only a half of the more complete and symmetrical space derived by taking into account the obverses of points as well as the points themselves. It seemed right, however, to call attention to the fact that space could be so regarded, as such fact is plainly brought out by the preceding results.

On the Square of Euler's Series.

By J. W. L. Glaisher, Sc.D., F.R.S.

[*Read June 13th*, 1889.]

Introduction, §§ 1–3.

1. Few results in pure algebra are more curious than Euler's celebrated theorem that the expanded value of the infinite product

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ \ldots$$

is the series

$$1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\&c.,$$

the general term being $(-1)^n q^{\frac{1}{2}n(3n\pm1)}$. The exponents are the pentagonal numbers, and the signs of the terms, after the first, are negative and positive in pairs.

Euler discovered the law of the series by actually multiplying together the factors $1-q$, $1-q^2$, $1-q^3$, &c. He published his result in 1750, and applied it to obtain various remarkable theorems relating to partitions, sums of divisors, &c.* Finally, in 1754, he succeeded in demonstrating the equality of the product and the series.†

2. Among the numerous and wonderful algebraical identities that sprang out from the analysis of Elliptic Functions as Jacobi developed the subject in the historic *Fundamenta Nova*, not only did the expanded value of the Eulerian product present itself, but also a hitherto unsuspected expression of its cube‡. The series thus found by Jacobi was

$$1-3q+5q^3-7q^6+9q^{10}-11q^{15}+13q^{21}-\&c.,$$

the general term being $(-1)^n(2n+1)q^{\frac{1}{2}n(n+1)}$. The exponents are the triangular numbers, and the terms are alternately positive and negative.

We thus see that the cube of the Eulerian product

$$(1-q)(1-q^2)(1-q^3)\ldots,$$

or of the Eulerian series $1-q-q^2+q^5+q^7-\&c.,$

is the Jacobian series $1-3q+5q^3-7q^6+\&c.$

3. The object of the present paper is to consider the law of the coefficients in the series which is equal to the square of the Eulerian product or series. I had no hope that these coefficients would follow any simple law, as in the Eulerian or Jacobian series; for, if such a law existed, it could not fail to have been discovered long ago by observation. Nevertheless, it seemed interesting to examine in some detail the functions of n which form the coefficients of the powers of q.

It will be seen in the following articles that the coefficient of q^n depends upon a function of $12n+1$, which is denoted by $G(12n+1)$. This function (in common with a great many functions connected with the Theory of Numbers) possesses the property that, if p and r are relatively prime, $G(pr)=G(p)\,G(r)$. We are thus enabled to obtain rules by means of which the value of G can be calculated with great facility (§§ 9, 10). It is shown that the numerical value of the coefficient of q^n can always be assigned by means of the *real* divisors of $12n+1$ (§ 13).

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* *Commentationes Arithmeticæ Collectæ*, Vol. I., pp. 91, 151.

† *Ibid.*, Vol. I., p. 234. Full references to the subject of Euler's product are given in a note to Art. 129 of H. J. S. Smith's "Report on the Theory of Numbers" (*Report of the British Association*, 1865, p. 345). See also §§ 33, 34 of the present paper.

‡ *Fundamenta Nova* (1829), p. 185.

In the latter part of the paper, various recurring formulæ, which are available for the calculation of G, are investigated. Two other functions E and H are also considered (§§ 15 and 28).

Squaring the Series, §§ 4–7.

4. Let

$$(1-q-q^3+q^5+q^7-q^{12}-q^{15}+\&c.)^2 = 1+P_1q+P_2q^2+P_3q^3+P_4q^4+\&c$$

Put q^{24} for q, and multiply throughout by q^3; we thus find

$$(q-q^{25}-q^{49}+q^{121}+q^{169}-q^{289}-q^{361}+\&c.)^2$$
$$= q^3+P_1q^{27}+P_2q^{51}+P_3q^{75}+P_4q^{99}+\&c.$$

This equation shows that P_n is equal to the number of compositions of $24n+2$ as the sum of two squares both of which are of the form $(12r\pm1)^2$, or of the form $(12r\pm5)^2$, diminished by the number of compositions in which one square is of one form and the other square of the other form.

5. If a number be expressible as the sum of two uneven squares, it must be $\equiv 2$, mod. 8; and if we put

$$8n+2 = (2r+1)^2+(2s+1)^2,$$

we have
$$4n+1 = (r-s)^2+(r+s+1)^2,$$

in which one square is even and the other uneven.

It can also be shown that every composition of $8n+2$ corresponds to four representations of $4n+1$.

6. By considering the separate cases which arise in § 4, it will be found that every composition of $24n+2$ of either of the forms

$$(12r\pm1)^2+(12s\pm1)^2, \quad (12r\pm5)^2+(12s\pm5)^2$$

corresponds to four representations of $12n+1$ of the form

$$(6r)^2+(6s\pm1)^2,$$

and that every composition of the form

$$(12r\pm1)^2+(12s\pm5)^2$$

corresponds to four representations of the form

$$(6r\pm2)^2+(6s\pm3)^2.$$

We are thus enabled to express the value of P_n by means of the representations of $12n+1$, instead of by means of the compositions of

$24n+2$; viz., we have found that $4P_n$ is equal to the number of representations of $12n+1$ as a sum of two squares of the form

$$(6r)^2+(6s\pm 1)^2$$

diminished by the number of representations of the form

$$(6r\pm 2)^2+(6s\pm 3)^2.*$$

7. We therefore put

$$P_n = G(12n+1),$$

where $4G(n)$ denotes the excess of the number of representations of n of the form $(6r)^2+(6s+1)^2$ over those of the form $(6r+2)^2+(6s+3)^2$. This excess may, of course, be negative.

We are only concerned with the function $G(n)$ for values of n which are $\equiv 1$, mod. 12, and in this case the representations of n as a sum of two squares are necessarily of the forms

$$(6r)^2+(6s+1)^2 \text{ or } (6r+2)^2+(6s+3)^2.$$

In what follows, the argument of the function G will always be supposed to be $\equiv 1$, mod. 12.

The Function G, §§ 8–14.

8. If p and q are any two numbers $\equiv 1$, mod. 12, which are prime to each other, then

$$G(pr) = G(p)\,G(r).$$

To prove this, resolve p and r in all possible ways into their conjugate complex factors, so that

$$p = (a_1+ib_1)(a_1-ib_1) = (a_2+ib_2)(a_2-ib_2) = \dots,$$
$$r = (\alpha_1+i\beta_1)(\alpha_1-i\beta_1) = (\alpha_2+i\beta_2)(\alpha_2-i\beta_2) = \dots.$$

Let these equations be written

$$p = p_1p_1' = p_2p_2' = \dots = p_ap_a' = \pi_1\pi_1' = \pi_2\pi_2' = \dots = \pi_a\pi_a',$$
$$r = r_1r_1' = r_2r_2' = \dots = r_br_b' = \rho_1\rho_1' = \rho_2\rho_2' = \dots = \rho_\beta\rho_\beta',$$

where p_1 and p_1', p_2 and p_2', &c. are pairs of conjugate factors; and $p_1, p_2, \dots, p_a, r_1, r_2, \dots r_b$ are of the form $6r+i(6s+1)$, and $\pi_1, \pi_2, \dots \pi_a$, $\rho_1, \rho_2, \dots, \rho_\beta$ of the form $6r+2+i(6s+3)$.

Thus $\qquad G(p) = a-a, \quad G(r) = b-\beta.$

Since p and r are prime to each other, the resolutions of pr into pairs

* We may dispense with the double sign in $(6s\pm 1)^2$, $(6r\pm 2)^2$, $(6s\pm 3)^2$, if we admit negative as well as positive values of the integers r and s. This will be supposed to be the case in future, and the double sign will be omitted throughout the whole of the subsequent portion of the paper.

of conjugate complex factors will be of the four types

$$pr\,p'r',\quad p\rho\,p'\rho',\quad \pi r\,\pi' r',\quad \pi\rho\,\pi'\rho'.$$

Now the product of two numbers of the form $6r+i\,(6s+1)$, or of two numbers of the form $6r+2+i\,(6s+3)$, is of the form $6r+i\,(6s+1)$; and the product of two numbers of which one is of one form, and the other of the other form, is of the form $6r+2+i\,(6s+3)$.

Thus the numbers which are of the types pr and $\pi\rho$ are of the form $6r+i\,(6s+1)$, and those which are of the types $p\rho$, πr are of the form $6r+2+i\,(6s+3)$. Now the numbers of numbers of the forms pr, $p\rho$, πr, $\pi\rho$ are respectively ab, $a\beta$, ab, $a\beta$; therefore

$$\begin{aligned} G(pr) &= ab-a\beta-ab+a\beta \\ &= (a-\alpha)(b-\beta) = G(p)\,G(r).* \end{aligned}$$

9. The values of $G(n)$ for all values of n may therefore be derived by simple multiplication from the values of $G(n)$ for the cases in which n is a prime or a power of a prime. It is only necessary, therefore, to consider the values of $G(n)$ for these values of n.

The only possible factors of a number which is $\equiv 1$, mod. 12, are necessarily $\equiv 11$, 7, 5, or 1, mod. 12. The cases in which n is a prime, or a power of a prime, having one of these forms will now be examined.

(1) Values of $G(a^\alpha)$ where a is a prime $\equiv 11$ or 7, mod. 12. Primes of these forms have no complex factors. If α be uneven, $G(a^\alpha)=0$; for a^α cannot be expressed as a sum of two squares. If α be even,

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* It will be observed that the reasoning in the text admits of a much more general application. For suppose that, connected with a number P, we have a set of quantities $p_1, p_2, \dots, p_a, \pi_1, \pi_2, \dots \pi_\alpha$, and connected with a number R we have a similar set $r_1, r_2, \dots r_b, \rho_1, \rho_2, \dots \rho_\beta$; and suppose that the corresponding set, connected with the number PR, may be obtained by multiplying all the members of the P-set by those of the R-set. Suppose, further, that the quantities denoted by the Italic letters belong to a class A, and that the quantities denoted by the Greek letters belong to a class B, and that the product of two members of class A, or of class B, is a member of class A, but that the product of one member of class A and one member of class B is a member of class B. Then, if

$$\phi(P) = p_1^k + p_2^k + \dots + p_a^k \pm (\pi_1^k + \pi_2^k + \dots + \pi_\alpha^k),$$

and

$$\phi(R) = r_1^k + r_2^k + \dots + r_b^k \pm (\rho_1^k + \rho_2^k + \dots + \rho_\beta^k),$$

it follows that

$$\phi(PR) = \phi(P)\,\phi(R).$$

The function G belongs to the particular case of $k=0$, the negative sign being taken. As other examples of the application of this reasoning may be mentioned the function $E_k(n)$ which expresses the excess of the sum of the k^{th} powers of the divisors of n which $\equiv 1$, mod. 4, over the sum of the k^{th} powers of the divisors which $\equiv 3$, mod. 4; and the function $H_k(n)$ which expresses the sum of the k^{th} powers of the divisors of n which $\equiv 1$, mod. 3, over the sum of the k^{th} powers of the divisors which $\equiv 2$, mod. 3. The function $E(n)$ and $H(n)$ (§§ 15 and 28) are the particular cases corresponding to $k=0$ of the functions $E_k(n)$ and $H_k(n)$.

$G(a^\alpha) = 1$; for p^α can be expressed as a sum of two squares in one way, *i.e.*, as $0^2 + (a^{\frac{1}{2}\alpha})^2$. This partition* is of the form $(6r)^2 + (6s+1)^2$; and it gives rise to four representations; therefore the value of G is $+1$.

(2) Values of $G(b^\beta)$ where b is a prime $\equiv 5$, mod. 12.

In this case the two conjugate complex factors of b are of the form $6r+2 \pm i(6s+1)$. Thus $G(b) = 0$. The partitions of b^2 as the sum of two squares consist of $0^2 + b^2$ and a single partition of the form $(6r+2)^2 + (6s+3)^2$. The former gives rise to four representations, the latter to eight. Thus $G(b^2) = 1-2 = -1$.

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* In counting the number of compositions of a number as a sum of squares, the order of the squares is to be taken into account. In counting the representations, the signs of the roots of the squares are to be taken into account as well. Each composition of a number as a sum of r squares none of which is zero gives rise, therefore, to 2^r representations. When we are restricting ourselves to uneven squares, it is convenient to consider compositions; but, when even squares are admitted (including zero among them), it is necessary to consider representations instead. There seems to be need of a word to express a partitionment into squares without reference to their order or to the signs of their roots. For this purpose I have in other writings used the word *resolution*; but in this paper I use the word *partition*, which seems to me preferable. Thus, for example, taking two squares, the partition $1^2 + 2^2$ gives rise to the two compositions $1^2 + 2^2$ and $2^2 + 1^2$, and to the eight representations

$$(+1)^2 + (+2)^2, \quad (-1)^2 + (+2)^2, \quad (+1)^2 + (-2)^2, \quad (-1)^2 + (-2)^2,$$
$$(+2)^2 + (+1)^2, \quad (-2)^2 + (+1)^2, \quad (+2)^2 + (-1)^2, \quad (-2)^2 + (-1)^2;$$

the partition $1^2 + 0^2$ gives rise to two compositions, but to only four representations. (The square 0^2 is treated in exactly the same manner as any other square, as far as partitions or compositions are concerned; but, when representations are considered, it differs from all other squares in having one root instead of two.)

It seems to me convenient to regard these meanings of the words *partition* and *composition* as of general application, whenever a number is to be partitioned into a given number of the parts $\alpha, \beta, \gamma, \ldots$. Thus every distinct manner in which a number n can be produced by the addition of the parts $\alpha, \beta, \gamma, \ldots$ is called a *partition* into these parts. If, in addition, we take cognisance of the places occupied by the parts, we use the word *composition*. Thus two partitions are identical if the parts occurring in them are the same; but for two compositions to be the same, it is further necessary that the same part should occupy the same place. (We may, of course, consider partitions in which no limit is placed on the number of times that any part may occur, or in which the same part may not occur twice, or in which any other condition is imposed. When the word *partition* is used without qualification, it is understood that the same part may occur any number of times.) These definitions of partitions and compositions do not conflict in any respect with ordinary usage.

Partitions and compositions have reference solely to the magnitude of the parts by the addition of which the given number is produced. The word *representation* has reference to a different kind of problem, *i.e.*, to the number of possible solutions, in integral numbers, of systems of equations. Partitions and compositions are concerned only with the magnitudes of $\alpha, \beta, \gamma \ldots$, not with their structure; the word representation has the technical meaning assigned to it by Gauss in the Theory of Forms. I may add that the above use of partition corresponds exactly to Gauss's definition of *discerptio*; for he distinguishes between *discerptiones* and *repræsentationes* as follows:—"Discerptiones numerorum (ut formarum binariarum supra) in tria quadrata a repræsentationibus per $xx+yy+zz$ ita distinguimus, ut in illis ad solam quadratorum magnitudinem, in his vero insuper ad ipsorum ordinem radicumque signa respiciamus, adeoque repræsentationes $x = a$, $y = b$, $z = c$ et $x = a'$, $y = b'$, $z = c'$ pro diversis habeamus nisi simul $a = a'$, $b = b'$, $c = c'$; discerptiones autem in $aa + bb + cc$ et in $a'a' + b'b' + c'c'$ pro una, si nullo ordinis respectu habito hæc quadrata illis æqualia sunt" (*Disq. Arith.*, § 292).

In the case of b^3 there is no partition into two squares of the forms considered. Thus $G(b^3) = 0$. For b^4 we have the partitions of b^2 each multiplied by b^2, and a new partition, which is of the form

$$(6r)^2 + (6s+1)^2.$$

Thus $$G(b^4) = 1-2+2 = 1.$$

Proceeding in this manner, we find that, if β be uneven, $G(b^\beta) = 0$, and that, if β be even, $G(b^\beta) = (-1)^{\frac{1}{2}\beta}$.

(3) Values of $G(c^\gamma)$ where c is a prime $\equiv 1$, mod. 12.

The two conjugate complex factors of c may be either of the form $6r \pm i(6s+1)$ or of the form $6r+2 \pm i(6s+3)$. Consider these two cases separately.

(i.) If c be of the form $(6r)^2+(6s+1)^2$, then

$$G(c) = 2, \quad G(c^2) = 3, \quad G(c^3) = 4,$$

and in general $$G(c^\gamma) = \gamma+1.$$

(ii.) If c be of the form $(6r+2)^2+(6s+3)^2$, then

$$G(c) = -2, \quad G(c^2) = 3, \quad G(c^3) = -4,$$

and in general $$G(c^\gamma) = (-1)^\gamma(\gamma+1).$$

10. Thus, on the whole, we find that, if

$$12n+1 = a_1^{\alpha_1} a_2^{\alpha_2} \dots b_1^{\beta_1} b_2^{\beta_2} \dots e_1^{\epsilon_1} e_2^{\epsilon_2} \dots f_1^{\phi_1} f_2^{\phi_2},$$

where

$a_1, a_2, \dots$ are primes $\equiv$ 11 or 7, mod. 12,

$b_1, b_2, \dots$ „ „ $\equiv$ 5, mod. 12,

$e_1, e_2, \dots$ „ „ $\equiv$ 1, mod. 12, and of the form $(6r)^2+(6s+1)^2$,

$f_1, f_2, \dots$ „ „ $\equiv$ 1, mod. 12, and of the form $(6r+2)^2+(6s+3)^2$,

then $$G(12n+1) = 0,$$

unless $\alpha_1, \alpha_2, \dots$, and $\beta_1, \beta_2, \dots$, are all even (including zero as an even number); and that, if these exponents are all even, then

$$G(12n+1) = (-1)^{\frac{1}{2}\beta_1+\frac{1}{2}\beta_2+\dots} \times (\epsilon_1+1)(\epsilon_2+1)\dots \times (-1)^{\phi_1+\phi_2+\dots}(f_1+1)(f_2+1)\dots.$$

For example,

$$P_{14} = G(169) = G(13^2) = +3,$$

$$P_{22} = G(265) = G(5 \cdot 53) = G(5)\,G(53) = 0,$$

$$P_{27} = G(325) = G(5^2 \cdot 13) = G(5^2)\,G(13) = -1 \times -2 = +2,$$

$$P_{79} = G(949) = G(13 \cdot 73) = G(13)\,G(73) = -2 \times -2 = +4.$$

As a more complicated example, we may take

$$G(11^4 . 19^2 . 5^3 . 17^4 . 13^3 . 37 . 97^3 . 109^4),$$

which $$= -1 \times 3 \times 2 \times -4 \times 5 = +120.$$

The even powers of the primes 11 and 19 (which are $\equiv 11$ and 7, mod. 12) produce no effect. The same is true of the evenly even power of 17 (which $\equiv 5$, mod. 12), but the unevenly even power of 5 (which also $\equiv 5$, mod. 12) gives rise to the factor -1. The only primes remaining are $\equiv 1$, mod. 12. The first two, 13 and 37, are of the form $(6r)^2+(6s+1)^2$; they give rise to the factors 3 and 2. The last two are of the form $(6r+2)^2+(6s+3)^2$, and give rise to the factors $(-1)^3 4$ and $(-1)^4 5$.

11. It will be noticed that for a great many values of n the value of P_n will be zero; for this will happen whenever any of the prime factors of $12n+1$ which $\equiv 11$, 7, or 5, mod. 12, occur with uneven exponents.

It is only in the case of prime factors which are $\equiv 1$, mod. 12, and when these factors occur with uneven exponents, that we have to take into consideration their complex factors, or, which is the same thing, resolve them into the sum of two squares. There is but one such partition for each prime, and upon its nature depends the sign which is to be attributed to the factor to which it gives rise in the value of G.

12. We may conveniently use the term *character* to distinguish between the cases when the prime is of the form $(6r)^2+(6s+1)^2$, and when it is of the form $(6r+2)^2+(6s+3)^2$. In the former case the character will be said to be positive; in the latter, negative.

Thus the character of a prime $\equiv 1$, mod. 12, is positive or negative, according as, when expressed as a sum of two squares, it is the even or the uneven square which is divisible by 3, or, which is the same thing, according as it is the even or uneven term in the complex factors of the prime which is divisible by 3.

We may also extend the idea of character to partitions, compositions, or representations, which will be distinguished as positive or negative, according as it is the even or the uneven square which is divisible by 3.

13. It is singular that the determination of the magnitude of P_n should depend wholly on the real factors of $12n+1$, and that it is only for the sake of the sign that we have to attend to the complex factors. Even for this purpose, it is only occasionally that recourse

to the complex factors is necessary, and, when such is the case, it is only the character of certain prime factors that has to be determined.

14. I give below a table of the values of the coefficient

$$P_n = G(12n+1),$$

for all non-zero values of n up to $n = 100$. The values of the function $E(12n+1)$, which forms the subject of §§ 15–18, are also added.

TABLE I.—Values of $P_n = G(12n+1)$, and of $E(12n+1)$, for all values of n for which P_n is not zero from $n = 1$ to $n = 100$.

n	$12n+1$	$P_n = G(12n+1)$	$E(12n+1)$	n	$12n+1$	$P_n = G(12n+1)$	$E(12n+1)$
1	13	−2	2	48	577	+2	2
2	25	−1	3	50	601	+2	2
3	37	+2	2	51	613	+2	2
4	49	+1	1	52	625	+1	5
5	61	+2	2	53	637	−2	2
6	73	−2	2	55	661	+2	2
8	97	−2	2	56	673	+2	2
9	109	−2	2	59	709	−2	2
10	121	+1	1	61	733	−2	2
13	157	+2	2	63	757	−2	2
14	169	+3	3	64	769	+2	2
15	181	−2	2	66	793	−4	4
16	193	+2	2	69	829	−2	2
19	229	−2	2	70	841	−1	3
20	241	−2	2	71	853	+2	2
23	277	−2	2	73	877	+2	2
24	289	−1	3	77	925	−2	6
26	313	+2	2	78	937	+2	2
27	325	+2	6	79	949	+4	4
28	337	−2	2	80	961	+1	1
29	349	+2	2	83	997	+2	2
30	361	+1	1	84	1009	−2	2
31	373	+2	2	85	1021	+2	2
33	397	+2	2	86	1033	−2	2
34	409	−2	2	89	1069	+2	2
35	421	−2	2	91	1093	−2	2
36	433	+2	2	93	1117	−2	2
38	457	−2	2	94	1129	−2	2
40	481	−4	4	96	1153	−2	2
44	529	+1	1	100	1201	+2	2
45	541	−2	2				

The next table shows the character of all primes $\equiv 1$, mod. 12, up to 12373. By its means we are able to write down at once (§ 10) the values of P_n as far as $n = 1031$, and, unless $12n+1$ be prime, to a very much greater extent. The manner in which the table was obtained is explained in § 19.*

TABLE II. — Showing the positive or negative character of the primes $\equiv 1$, mod. 12, up to 12373.

13 −	997 +	2017 −	3169 +	4297 +	5689 −
37 +	1009 −	2029 −	3181 −	4357 +	5701 −
61 +	1021 +	2053 +	3217 −	4441 +	5737 −
73 −	1033 −	2089 −	3229 −	4513 +	5749 −
97 −	1069 +	2113 −	3253 −	4549 +	5821 −
109 −	1093 −	2137 +	3301 +	4561 +	5857 −
157 +	1117 −	2161 −	3313 −	4597 +	5869 −
181 −	1129 −	2221 −	3361 −	4621 +	5881 −
193 +	1153 −	2269 +	3373 −	4657 −	5953 −
229 −	1201 +	2281 −	3433 −	4729 −	6037 +
241 −	1213 −	2293 +	3457 −	4789 +	6073 +
277 −	1237 −	2341 −	3469 −	4801 +	6121 −
313 +	1249 −	2377 −	3517 +	4813 +	6133 +
337 −	1297 +	2389 +	3529 +	4861 −	6217 −
349 +	1321 +	2437 +	3541 +	4909 −	6229 +
373 +	1381 −	2473 +	3613 +	4933 −	6277 +
397 +	1429 +	2521 +	3637 −	4957 −	6301 −
409 −	1453 −	2557 −	3673 +	4969 +	6337 +
421 −	1489 −	2593 +	3697 +	4993 −	6361 −
433 +	1549 +	2617 −	3709 +	5077 +	6373 +
457 −	1597 −	2677 −	3733 −	5101 −	6397 +
541 −	1609 −	2689 −	3769 +	5113 +	6421 −
577 +	1621 −	2713 −	3793 −	5197 +	6469 −
601 +	1657 +	2749 +	3853 −	5209 +	6481 −
613 +	1669 −	2797 −	3877 +	5233 +	6529 +
661 +	1693 +	2833 +	3889 +	5281 +	6553 +
673 +	1741 +	2857 −	4021 −	5413 −	6577 −
709 −	1753 −	2917 +	4057 +	5437 −	6637 +
733 −	1777 −	2953 +	4093 −	5449 +	6661 −
757 −	1789 +	3001 −	4129 +	5521 +	6673 −
769 +	1801 +	3037 +	4153 +	5557 −	6709 +
829 −	1861 +	3049 −	4177 −	5569 −	6733 −
853 +	1873 −	3061 +	4201 −	5581 +	6781 −
877 +	1933 +	3109 +	4261 +	5641 −	6793 +
937 +	1993 +	3121 −	4273 −	5653 +	6829 +

* Barlow's Tables of 1814 contain the complete resolutions of all numbers into their prime factors up to 10,000. Chernac (*Cribrum Arithmeticum*, 1811) gives all the prime factors of numbers up to 1,020,000. The Tables of Burckhardt, Dase, and J. Glaisher give the least factors of numbers up to 9,000,000.

6841 −	7681 +	8581 +	9433 −	10429 +	11497 +
6949 −	7717 −	8629 +	9601 +	10453 +	11593 +
6961 −	7741 −	8641 +	9613 −	10477 −	11617 +
6997 −	7753 −	8677 −	9649 −	10501 +	11677 −
7057 +	7789 +	8689 −	9661 −	10513 +	11689 +
7069 −	7873 −	8713 −	9697 −	10597 +	11701 −
7129 −	7933 +	8737 +	9721 −	10657 −	11821 +
7177 +	7993 +	8761 −	9733 +	10729 −	11833 +
7213 +	8017 +	8821 +	9769 −	10753 +	11941 +
7237 −	8053 −	8893 +	9781 +	10789 +	11953 +
7297 −	8089 +	8929 +	9817 −	10837 +	12037 −
7309 +	8101 +	8941 +	9829 −	10861 −	12049 −
7321 +	8161 −	9001 −	9901 −	10909 +	12073 +
7333 −	8209 +	9013 −	9949 +	10957 −	12097 +
7369 +	8221 +	9049 −	9973 −	10993 −	12109 −
7393 +	8233 +	9109 +	10009 −	11113 +	12157 −
7417 +	8269 +	9133 −	10069 −	11149 −	12241 +
7477 −	8293 +	9157 +	10093 −	11161 −	12253 +
7489 −	8317 +	9181 +	10141 +	11173 +	12277 +
7537 +	8329 −	9241 +	10177 +	11197 +	12289 +
7549 +	8353 −	9277 −	10273 −	11257 −	12301 −
7561 −	8377 −	9337 +	10321 +	11317 −	12373 +
7573 −	8389 +	9349 +	10333 −	11329 +	
7621 −	8461 +	9397 +	10357 +	11353 −	
7669 −	8521 +	9421 −	10369 −	11437 −	

Connexion with the Function $E(n)$, §§ 15–18.

15. The function $E(n)$, which may be defined as the excess of the number of divisors of n which are $\equiv 1$, mod. 4, over the number of divisors which are $\equiv 3$, mod. 4, has been considered in Vol. XV., pp. 104–122.* If n be uneven, $E(n)$ is equal to the number of primary complex numbers having n as their norm; and for all values of n, $4E(n)$ is equal to the number of representations of n as a sum of two squares.

Thus, when $n \equiv 1$, mod. 12, $4E(n)$ is equal to the sum of the numbers of positive and negative representations of n, while $4G(n)$ is numerically equal to their difference. When, therefore, all the compositions of n are of the same character, $G(n)$ is numerically equal to $E(n)$. This will evidently be the case when n is a prime, and, as will be shown in the next article, it will happen also whenever n contains no prime factor which $\equiv 5$, mod. 12.

* "On the Function which denotes the difference between the number of $(4m+1)$-divisors, and the number of $(4m+3)$-divisors of a Number," read February 14th, 1884.

The function $E(n)$ satisfies the condition that, if p and r are prime to each other, $E(pr) = E(p)\,E(r)$; and, in general, if

$$n = a_1^{\alpha_1} a_2^{\alpha_2} \ldots b_1^{\beta_1} b_2^{\beta_2} \ldots,$$

where $a_1, a_2, \ldots$ are primes $\equiv 3$, mod. 4,

and $b_1, b_2, \ldots$,, $\equiv 1$, mod. 4,

then $E(n) = 0$ unless $\alpha_1, \alpha_2, \ldots$ are all even (including zero as an even number), and if these exponents are all even,

$$E(n) = (\beta_1+1)(\beta_2+1)\ldots.$$

16. In the calculation, therefore, of E and G, in cases in which they do not both vanish, prime factors which are $\equiv 11$ or 7, mod. 12, give rise to the factor unity, that is to say, they produce no effect; prime factors which are $\equiv 1$, mod. 12, give rise to the same factors in E and G, though they may differ in sign. But in the case of prime factors which are $\equiv 5$, mod. 12, there is a difference of numerical value; for, if b be a prime of this form, the factor b^β occurring in n gives rise to the factor $\beta+1$ in E, but in G it gives rise to the factors -1 or 0, according as β is even or uneven.

Thus (supposing n, as always, to be $\equiv 1$, mod. 12) the numerical values of $G(n)$ and $E(n)$ are the same whenever n contains no prime factor $\equiv 5$, mod. 12. If any such factor occurs raised to an uneven power, it reduces G to zero; if raised to an even power, it merely produces a change of sign. In E it gives rise in each case to a finite factor greater than unity.

The greater number of zero values of $G(n)$ than of $E(n)$ is due to the presence of uneven powers of primes $\equiv 5$, mod. 12; viz., any one of the factors $5, 5^3, 5^5, \ldots, 17, 17^3, \ldots 29, 29^3, \ldots$ &c. reduces G to zero. For non-vanishing values of $G(n)$, we have seen that the only case in which a difference of numerical value occurs is when these factors present themselves with even exponents, and if $b_1, b_2, \ldots$ denote the prime factors of this form, and $\beta_1, \beta_2, \ldots$ are their exponents (supposed to be all even), then

$$G(n) = \pm \frac{E(n)}{(\beta_1+1)(\beta_2+1)\ldots}.$$

17. In Table I. (p. 190) the values of $E(n)$ were given in an additional column for the sake of comparison. Within the limits of that table there are seventeen arguments for which E remains finite while

G vanishes. These are—

$85 = 5 \cdot 17$,	$505 = 5 \cdot 101$,	$901 = 17 \cdot 53$,
$145 = 5 \cdot 29$,	$565 = 5 \cdot 113$,	$985 = 5 \cdot 197$,
$205 = 5 \cdot 41$,	$685 = 5 \cdot 137$,	$1105 = 5 \cdot 13 \cdot 17$,
$265 = 5 \cdot 53$,	$697 = 17 \cdot 41$,	$1165 = 5 \cdot 233$,
$445 = 5 \cdot 89$,	$745 = 5 \cdot 149$,	$1189 = 29 \cdot 41$.
$493 = 17 \cdot 29$,	$865 = 5 \cdot 173$,	

For the argument 1105 the value of E is 8; in all the other cases it is 4.*

There are six cases in which G does not vanish, but in which its numerical value differs from E. These are—

	Value of G.	Value of E.
$25 = 5^2$,	-1,	3,
$289 = 17^2$,	-1,	3,
$325 = 5^2 \cdot 13$,	2,	6,
$625 = 5^4$,	1,	5,
$841 = 29^2$,	-1,	3,
$925 = 5^2 \cdot 37$,	-2,	6.

It will be noticed that the corresponding values of G and E are connected by the relation given at the end of the preceding article.

18. The values of $G(n)$, in Table I., were originally calculated by finding the partitions of $24n+2$ into two uneven squares (§ 4), before I had obtained the method described in §§ 6–10. The corresponding values of $E(n)$ were taken from the paper in Vol. xv. of the *Proceedings*, already cited in the note to § 15. The table of $E(n)$ contained in that paper gives the values of $E(n)$ for all values of n up to $n = 1000$ for which $E(n)$ does not vanish.†

The Function $\chi(n)$, § 19.

19. In Vol. xx. (1884) of the *Quarterly Journal*‡ I have considered

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* It is evident that, when G vanishes and E does not, the value of E must necessarily be an evenly even number, for there must be an equal number of positive and negative compositions, each of which counts in E as $+2$.

† The definition of $E(n)$ applies to all positive integral values of n. The definition of $G(n)$ only applies to numbers $\equiv 1$, mod. 12.

‡ "On the Function $\chi(n)$," pp. 97–167.

the function $\chi(n)$, which is defined as the sum of the primary numbers having n as their norm.

If n be a prime $\equiv 1$, mod. 12, then either

$$n = (6r+1)^2+6s^2$$

or

$$n = (6r+3)^2+(6s+2)^2.$$

In the former case the value of $\chi(n)$ is

$$(-1)^{s(r+s)}(12r+2),$$

and in the latter

$$(-1)^{s(r+s)}(12r+6).$$

Therefore, when the character of a prime is positive, $\chi(n)$ is not divisible by 3; and when the character is negative, $\chi(n)$ is divisible by 3.

In the paper on $\chi(n)$ to which reference has just been made, I have given a table of the values of $\chi(n)$ for all primes $\equiv 1$, mod. 4, up to $n = 12377$. Table II. was derived from this table by selecting from it the primes $\equiv 1$, mod. 12, and affixing the sign $-$ or $+$ according as $\chi(n)$ was, or was not, divisible by 3.

Analytical Formulæ connected with the Functions E and G, §§ 20–27.

20. It can be shown by elliptic functions that, if n denote any number (and therefore $2n$ any even number), and m any uneven number,

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}^2 = 1+4\Sigma_1^{\infty} E(n)\, q^n,$$

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{4n^2} = 2\Sigma_0^{\infty} E(4n+1)\, q^{4n+1},$$

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}^2 = 4\Sigma_0^{\infty} E(4n+1)\, q^{8n+2}.$$

The second formula shows that the number of representations of $4n+1$ as the sum of an even and an uneven square is equal to $4E(4n+1)$, and the third formula shows that the number of representations of $8n+2$ as a sum of two uneven squares is also equal to $4E(4n+1)$. It follows, therefore, that if $n \equiv 1$, mod. 4, the number of representations of $2n$ as a sum of two squares is equal to the number of representations of n as a sum of two squares. This theorem, of which it is easy to give an arithmetical proof, has been referred to in § 5.

21. Evidently

$$(6r+1)^2+(6s)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+3)^2+(6s+2)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+1)^2+(6s+2)^2 \equiv 5, \text{ mod. } 12,$$

$$(6r+3)^2+(6s)^2 \equiv 9, \text{ mod. } 36.$$

By separating the terms whose exponents are $\equiv 1$ from those which are $\equiv 5$, mod. 12, in the second formula of the preceding article, we obtain the following results :—

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} E(12n+1)\, q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_0^{\infty} E(12n+5)\, q^{12n+5}.$$

We find also

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} = 2\,\Sigma_0^{\infty} E(36n+9)\, q^{36n+9};$$

but this formula is equivalent to the original equation, for, by replacing q by $q^{\frac{1}{4}}$, it becomes

$$\Sigma_{-\infty}^{\infty} q^{(2n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(2n)^2} = 2\,\Sigma_0^{\infty} E(36n+9)\, q^{4n+1}.$$

It is easy to see that, for all values of n,

$$E(36n+9) = E(4n+1);$$

for, if the highest power of 3 which occurs in $4n+1$ as a factor be uneven, E vanishes for both arguments, and if the highest power is even, $= 3^{2a}$ (including the case $a = 0$), then

$$E(4n+1) = E\left\{3^{2a}(4r+1)\right\} = E(3^{2a})\, E(4r+1) = E(4r+1);$$

and, similarly,

$$E(36n+9) = E\left\{3^{2a+2}(4r+1)\right\} = E(4r+1).$$

22. Treating in the same manner the third formula of § 20, by means of the congruences

$$(6r+1)^2+(6s+1)^2 \equiv 2, \quad \text{mod. } 24,$$

$$(6r+1)^2+(6s+3)^2 \equiv 10, \quad \text{mod. } 24,$$

$$(6r+3)^2+(6s+3)^2 \equiv 18, \quad \text{mod. } 36,$$

we obtain the formulæ

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} E(12n+1)\, q^{24n+2},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} = \Sigma_0^{\infty} E(12n+5)\, q^{24n+10}.$$

We find also

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+3)^2}\right\}^2 = 4\Sigma_0^{\infty} E(36n+9)\, q^{72n+18},$$

which is, however, only a repetition of the original formula.

23. It is worth while to notice the identical relations to which the formulæ in the two preceding articles give rise, viz.,

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+1)^2}\right\}^2,$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+3)^2}.$$

The original formulæ in § 20 give rise to the identity

$$\Sigma_{-\infty}^{\infty} q^{(2n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(2n)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(2n+1)^2}\right\}^2,$$

from which, of course, the two preceding identities might be directly obtained by the method employed in the two preceding articles.

24. Passing now to the function G, we have, from § 4,

$$\left\{\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} G(12n+1)\, q^{24n+2},$$

and from § 5 it follows that

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} - \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} G(12n+1)\, q^{12n+1}.$$

These formulæ correspond to

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} E(12n+1)\, q^{24n+2}$$

and

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} E(12n+1)\, q^{12n+1},$$

which were given in §§ 22 and 21.

25. Combining, by addition and subtraction, the first and third formulæ, we find

$$\left\{\Sigma_{-\infty}^{\infty} q^{(12n+1)^2}\right\}^2 + \left\{\Sigma_{-\infty}^{\infty} q^{(12n+5)^2}\right\}^2 = \tfrac{1}{2}\Sigma_0^{\infty} \left\{E(12n+1) + G(12n+1)\right\} q^{24n+2}$$

$$\Sigma_{-\infty}^{\infty} q^{(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(12n+5)^2} = \tfrac{1}{4}\Sigma_0^{\infty} \left\{E(12n+1) - G(12n+1)\right\} q^{24n+2}.$$

These equations express the theorems:

(i.) The number of compositions of a number p (necessarily $\equiv 2$, mod. 24) as the sum of two squares, which are both of the form $(12n+1)^2$, or both of the form $(12n+5)^2$, is equal to

$$\tfrac{1}{2}\left\{E(p) + G(p)\right\}.$$

(ii.) The number of compositions of a number p (necessarily $\equiv 2$, mod. 24) as the sum of two squares, of which one is of the form $(12n+1)^2$ and the other of the form $(12n+5)^2$, is equal to

$$\tfrac{1}{2}\left\{E(p) - G(p)\right\}.$$

26. Combining in the same manner the second and fourth formulæ of § 24, we find

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} = \tfrac{1}{2}\Sigma_0^{\infty} \{E(12n+1)+G(12n+1)\} q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_0^{\infty} \{E(12n+1)-G(12n+1)\} q^{12n+1},$$

which express the theorems:

(i.) The number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $(6r+1)^2+(6s)^2$ is equal to

$$2\{E(p)+G(p)\}.$$

(ii.) The number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $(6r+3)^2+(6s+2)^2$ is equal to

$$2\{E(p)-G(p)\}.*$$

27. It will be noticed that the formulæ in the last two articles lead to the identities

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+1)^2}\right\}^2 + \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+5)^2}\right\}^2,$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = 2\Sigma_{-\infty} q^{\frac{1}{2}(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+5)^2}.$$

The Function $H(n)$, §§ 28, 29.

28. It can be shown by the Theory of Numbers that the number of representations of any uneven number m by the form x^2+3y^2 is equal to four times the excess of the number of divisors of m which $\equiv 1$, mod. 3, over the number of divisors which $\equiv 2$, mod. 3.

Let $H(n)$ denote the excess of the number of divisors of n which $\equiv 1$, mod. 3, over those which $\equiv 2$, mod. 3.

If an uneven number be represented by the form x^2+3y^2, either x or y must be even and the other uneven; observing that

$$(2r+1)^2+3(2s)^2 \equiv 1, \text{ mod. } 4,$$

$$(2r)^2+3(2s+1)^2 \equiv 3, \text{ mod. } 4,$$

we thus obtain the analytical theorems,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = 2\Sigma_0^{\infty} H(4n+1) q^{4n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{4m^2} \times \Sigma_{-\infty}^{\infty} q^{3n^2} = 2\Sigma_0^{\infty} H(4n+3) q^{4n+3},$$

where n denotes any number, and m any uneven number.

* In (i.) the number p is expressed as a sum of two squares, of which the even square is divisible by 3, and the uneven square is not; in (ii.) p is expressed as a sum of two squares, of which the uneven square is divisible by 3, and the even square is not.

Taking the first formula, since

$$(6r+1)^2+3\,(2s)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+3)^2+3\,(2s)^2 \equiv 9, \text{ mod. } 12,$$

we find, by similar reasoning to that employed in § 21,

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} H(12n+1)\, q^{12n+1}.$$

29. The function $H(n)$ possesses the property that, if p and q are prime to each other,

$$H(pq) = H(p)\, H(q).$$

This may be proved by exactly the same methods as in the case of the corresponding theorem

$$E(pq) = E(p)\, E(q);$$

i.e., analytically, as in *Proc. Lond. Math. Soc.*, Vol. xv., pp. 104, 105, or by general reasoning, as in the note to § 8 of the present paper.

The former method shows that $H(n)$ vanishes unless every prime divisor of n which $\equiv 2$, mod. 3, is raised to an even power, and that if, in general, $n = 3^r u v^2$, where all the prime factors of u are $\equiv 1$, mod. 2, and all the prime factors of v are $\equiv 2$, mod. 3, then

$$H(n) = H(u) = \nu(u),$$

$\nu(u)$ denoting the number of divisors of u.

The Functions G, E, H, §§ 30–32.

30. We may obtain an equation similar in form to the last equation in § 28, but in which G is involved instead of H, by the following method.

$$\begin{aligned}\Sigma_1^{\infty} P_n q^n &= \Pi_1^{\infty}(1-q^n)^3 \\ &= \Pi_1^{\infty}(1-q^{2n})\,\Pi_1^{\infty}\left(\frac{1-q^n}{1+q^n}\right) \\ &= \Sigma_{-\infty}^{\infty}(-1)^n q^{n(3n+1)} \times \Sigma_{-\infty}^{\infty}(-1)^n q^{n^2}.\end{aligned}$$

Putting q^{12} for q, this equation becomes

$$\Sigma_1^{\infty} P_n q^{12n} = \Sigma_{-\infty}^{\infty}(-1)^n q^{12n(3n+1)} \times \Sigma_{-\infty}^{\infty}(-1)^n q^{12n^2};$$

whence, replacing P_n by $G(12n+1)$, and multiplying throughout by q, we find

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty}(-1)^n q^{12n^2} = \Sigma_0^{\infty}(-1)^n G(12n+1)\, q^{12n+1}.$$

31. It was found, in §§ 22 and 24, that

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} E(12n+1)\, q^{24n+2},$$

$$\left\{\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} G(12n+1)\, q^{24n+2},$$

and, in §§ 28 and 30, that

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} H(12n+1)\, q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} (-1)^n q^{12n^2} = \Sigma_0^{\infty} G(12n+1)\, q^{12n+1}.$$

Writing these formulæ at full length, they become

$$E(1)\, q^2 + E(13)\, q^{26} + E(25)\, q^{50} + E(37)\, q^{74} + \&c.$$
$$= (q + q^{25} + q^{49} + q^{121} + q^{169} + \&c.)^2,$$

$$G(1)\, q^2 + G(13)\, q^{26} + G(25)\, q^{50} + G(37)\, q^{74} + \&c.$$
$$= (q - q^{25} - q^{49} + q^{121} + q^{169} - \&c.)^2,$$

and

$$H(1)\, q + H(13)\, q^{13} + H(25)\, q^{25} + H(37)\, q^{37} + \&c.$$
$$= (q + q^{25} + q^{49} + q^{121} + q^{169} + \&c.)(1 + 2q^{12} + 2q^{48} + 2q^{108} + 2q^{192} + \&c.),$$

$$G(1)\, q + G(13)\, q^{13} + G(25)\, q^{25} + G(37)\, q^{37} + \&c.$$
$$= (q - q^{25} - q^{49} + q^{121} + q^{169} - \&c.)(1 - 2q^{12} + 2q^{48} - 2q^{108} + 2q^{192} - \&c.).$$

The first pair of formulæ show that $G(12n+1)$ must vanish when $E(12n+1)$ vanishes; for $4E(12n+1)$ is equal to the total number of representations of $12n+1$ as a sum of two squares neither of which is divisible by 3, so that, when $E(12n+1)$ vanishes, $12n+1$ does not admit of being so expressed.

Similarly, from the second pair of formulæ we see that $G(12n+1)$ must vanish when $H(12n+1)$ vanishes; for $4H(12n+1)$ is equal to the total number of representations of $12n+1$ as the sum of an uneven square not divisible by 3 and of the triple of an even square.

Now $E(n)$ vanishes unless all the prime factors of n which are $\equiv 3$, mod. 4, are raised to even powers; and $H(n)$ vanishes unless all the prime factors of n which are $\equiv 2$, mod. 3, are raised to even powers.

Thus either $E(n)$ or $H(n)$ vanishes (or both vanish) unless all the prime factors of n which are $\equiv 3, 5, 7$, or 11, mod. 12, are raised to even powers. It follows, therefore, that $G(12n+1)$ vanishes, unless all the prime factors of $12n+1$ which are $\equiv 3, 5, 7$, or 11, mod. 12, are raised to even powers. This result is included in the arithmetical investigation of § 9.

32. Five years ago (in 1884) I found, by the analytical process of the preceding article, that P_n was always zero unless all the prime factors of $12n+1$ which were $\equiv 3, 5, 7,$ or 11, mod. 12, were raised to even powers. The process did not show, however, that it might not happen that P_n should vanish, even when this condition was satisfied, and I did not then attempt an arithmetical investigation. It appears from § 9, however, that P_n can never vanish when all the prime factors of the above forms occur with even exponents. I have thought the results contained in the four preceding articles worth giving, partly because of the analytical proof which we thus obtain of the theorem that P_n vanishes unless the prime factors of $12n+1$ which are of a certain form occur with even exponents, and partly for the sake of introducing the function H, which belongs to the same class of coefficients as E and G.

Linear relations connecting the values of P_n, §§ 33–44.

33. When Euler had obtained the formula

$$(1-q)(1-q^2)(1-q^3)\ldots = 1-q-q^2+q^5+q^7-\&c.,$$

he applied it in the following manner to obtain results connected with the Theory of Numbers:—

(i.) If $P(n)$ denote the number of partitions of the number n into the parts 1, 2, 3, ..., repetitions not excluded, we have

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots} = 1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.,$$

whence it follows that

$$\{1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.\}\{1-q-q^2+q^5+q^7-\&c.\} = 1.$$

Equating the coefficients of q^n, we find, for all values of n,

$$P(n)-P(n-1)-P(n-2)+P(n-5)+P(n-7)-\&c. = 0,$$

the series being continued so long as the arguments remain positive.* The value of $P(0)$, when it occurs, is supposed to be unity.

(ii.) By taking logarithms and differentiating, we deduce from Euler's series the equation

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c. = \frac{q+2q^2-5q^5-7q^7+\&c.}{1-q-q^2+q^5+q^7-\&c.}.$$

* *Commentationes Arithmeticæ Collectæ*, Vol. I., p. 91.

Denoting by $\sigma(n)$ the sum of the divisors of n, this equation may be written

$$\{\sigma(1)q+\sigma(2)q^2+\sigma(3)q^3+\&c.\}\{1-q-q^2+q^5+q^7-\&c.\}$$
$$=q+2q^2-5q^5-7q^7+\&c.;$$

whence, by equating coefficients, we obtain the relation

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\&c.=0,$$*

if we adopt the convention that $\sigma(0)$, *i.e.*, $\sigma(n-n)$, when it occurs, is to have assigned to it the value n.

34. It may be remarked that, by writing the equation obtained in the preceding section in the form

$$\frac{q+2q^2-5q^5-7q^7+\&c.}{(1-q)(1-q^2)(1-q^3)\ldots}=\sigma(1)q+\sigma(2)q^2+\sigma(3)q^3+\&c.,$$

we find, by equating the coefficients of q^n on each side,

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c.=\sigma(n).$$†

This result, combined with Euler's second theorem, shows that the values of

$$\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\&c.$$

and

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c.$$

are equal.

35. We may apply Euler's second method to deduce from Jacobi's formula,

$$\{(1-q)(1-q^2)(1-q^3)\ldots\}^3=1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

a corresponding property of the function $\sigma(n)$.

For we thus find

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c.=\frac{q-5q^3+14q^6-30q^{10}+\&c.}{1-3q+5q^3-7q^6+9q^{10}-\&c.};$$

from which, by equating coefficients, we have

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+9\sigma(n-10)-\&c.=0,$$‡

if we assign to $\sigma(0)$, *i.e.* to $\sigma(n-n)$, the meaning $\frac{1}{2}n$.

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* *Ibid.*, pp. 151–154.

† *Messenger of Mathematics*, Vol. XII., p. 170. This paper also contains other theorems connecting partitions and sums of divisors.

‡ *Quarterly Journal*, Vol. XIX., p. 220. It is so obvious, however, that Euler's method is exactly applicable to Jacobi's formula, that the result must have been noticed before. See also *Proc. Camb. Phil. Soc.*, Vol. V., p. 109.

36. Passing now to the formula

$$\{(1-q)(1-q^2)(1-q^3)\ldots\}^2 = 1+P_1q+P_2q^2+P_3q^3+\&c.,$$

I proceed to consider the properties of the coefficient P_n that can be obtained by methods of a similar character to those employed in the three preceding sections.

(i.) Applying Euler's first process (§ 33), we have

$$\{1+P_1q+P_2q^2+P_3q^3+\&c.\}\{1+P(1)q+P(2)q^2+P(3)q^3+\&c.\}$$
$$= 1-q-q^2+q^5+q^7-\&c.;$$

whence it follows that

$$P_n+P(1)P_{n-1}+P(2)P_{n-2}+\ldots+P(n-1)P(1)+P(n)$$
$$= 0 \text{ or } (-1)^r,$$

according as n is not of the form $\frac{1}{2}r(3r\pm1)$, or is equal to $\frac{1}{2}r(3r\pm1)$.

(ii.) Applying Euler's second method, we have

$$2\left\{\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c.\right\} = -\frac{P_1q+2P_2q^2+3P_3q^3+\&c.}{1+Pq+P_2q^2+P_3q^3+\&c.},$$

which gives the relation

$$-\tfrac{1}{2}nP_n = \sigma(1)P_{n-1}+\sigma(2)P_{n-2}+\ldots+\sigma(n-1)P_1+\sigma(n).$$

37. Since

$$\{1+P(1)q+P(2)q^2+P(3)q^3+\&c.\}\{1-3q+5q^3-7q^6+9q^{10}-\&c.\}$$
$$= 1+P_1q+P_2q^2+P_3q^3+\&c.,$$

and

$$\{1+P_1q+P_2q^2+P_3q^3+\&c.\}\{1-q-q^2+q^5+q^7-\&c.\}$$
$$= 1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

we find, by equating coefficients,

$$P(n)-3P(n-1)+5P(n-3)-7P(n-6)+\&c. = P_n,$$

and

$$P_n-P_{n-1}-P_{n-2}+P_{n-5}+P_{n-7}-\&c.$$
$$= 0 \text{ or } (-1)^r(2r+1),$$

according as n is not of the form $\frac{1}{2}r(r+1)$, or is equal to $\frac{1}{2}r(r+1)$. The value of P_0 is supposed to be unity.

38. Since

$$(1-q-q^2+q^5+q^7-\&c.)^2 = 1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

we find, by differentiating,

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(q+2q^2-5q^5-7q^7+\&c.)$$
$$=q-5q^5+14q^6-30q^{10}+\&c.;$$

whence, by equating coefficients,

$$P_{n-1}+2P_{n-2}-5P_{n-5}-7P_{n-7}+\&c.=0 \quad \text{or} \quad (-1)^{r-1}\frac{r(r+1)(2r+1)}{6},$$

according as n is not of the form $\frac{1}{2}r(r+1)$, or is equal to $\frac{1}{2}r(r+1)$.*

In §§ 34, 35, and 36, three fractional formulæ have been obtained for the series $\Sigma_1^\infty \sigma(n)\,q^n$. We do not obtain new formulæ by equating these fractional expressions. For example, from §§ 35 and 36 we have

$$-\tfrac{1}{2}(P_1q+2P_2q^2+3P_3q^3+\&c.)(1-q-q^3+q^5+q^7-\&c.)$$
$$=q-5q^3+14q^6-30q^{10}+\&c.,$$

giving

$$nP_n-(n-1)P_{n-1}-(n-2)P_{n-2}+(n-5)P_{n-5}+(n-7)P_{n-7}-\&c.$$
$$=0 \quad \text{or} \quad (-1)^r\frac{r(r+1)(2r+1)}{3},$$

according as n is not of the form $\frac{1}{2}r(r+1)$, or is equal to $\frac{1}{2}r(r+1)$.

This formula, however, is easily deducible from the expressions for $P_{n-1}+2P_{n-2}-5P_{n-5}-\&c.$, and $P_n-P_{n-1}-P_{n-2}-\&c.$, which have been already obtained.

In the six following articles, various formulæ involving the functions P_n, and connecting them with other functions, are obtained by equating coefficients.

39. From § 30 we have

$$1+P_1q+P_2q^2+P_3q^3+\&c.$$
$$=(1-q^3-q^4+q^{10}+q^{14}-\&c.)(1-2q+2q^4-2q^9+2q^{16}-\&c.),$$

whence

$$\frac{1+P_1q+P_2q^2+P_3q^3+\&c.}{1-P_1q+P_2q^2-P_3q^3+\&c.}=\frac{1-2q+2q^4-2q^9+2q^{16}-\&c.}{1+2q+2q^4+2q^9+2q^{16}+\&c.}.$$

---

* In the case of the function $P(n)$, we have the theorem

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c.=\sigma(n),$$

which, however, only holds good when n is uneven (*Messenger*, Vol. XII., p. 170).

Multiplying up and equating the coefficients of q^n, we find that, if n be any uneven number,

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+2P_{n-16}+\&c.=0.*$$

40. By multiplying the first equation in the preceding article by $(1-q^3)(1-q^4)(1-q^6)\ldots$, we have

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1-q^3-q^4+q^{10}+q^{14}-\&c.)$$
$$=(1+P_1q^3+P_2q^4+P_3q^6+\&c.)(1-2q+2q^4-2q^9+2q^{16}-\&c.);$$

whence, by equating coefficients, we find, for all values of n,

$$P_{2n}-P_{2n-2}-P_{2n-4}+P_{2n-10}+P_{2n-14}-\&c.$$
$$=P_n+2P_{n-2}+2P_{n-8}+2P_{n-18}+\&c.,$$

and
$$P_m-P_{m-2}-P_{m-4}+P_{m-10}+P_{m-14}-\&c.$$
$$=-2P_n-2P_{n-4}-2P_{n-12}-2P_{n-24}-\&c.,$$

where, in the second formula, m denotes $2n+1$.

In the right-hand member of the first formula the suffix in the general term is $n-2r^2$, and in the second formula it is $n-2r^2-2r$.

41. If we denote by $Q(n)$ the number of partitions of n into the elements 1, 2, 3, ... in which no part is repeated in the same partition, so that

$$(1+q)(1+q^2)(1+q^3)\ldots$$
$$=1+Q(1)q+Q(2)q^2+Q(3)q^3+Q(4)q^4+\&c.,$$

then we have

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1-q^3-q^4+q^{10}+q^{14}-\&c.)$$
$$=\{1+Q(1)q+Q(2)q^2+Q(3)q^3+\&c.\}(1-3q+5q^3-7q^6+9q^{10}-\&c.);$$

whence, by equating coefficients,

$$P_n-P_{n-2}-P_{n-4}+P_{n-10}+P_{n-14}-\&c.$$
$$=Q(n)-3Q(n-1)+5Q(n-3)-7Q(n-6)+\&c.,$$

for all values of n; the value of $Q(0)$ being supposed to be unity.

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* Similar formulæ relating to the functions χ, E, and σ are given in the *Quarterly Journal*, Vol. xx., pp. 120, 121. A formula of the same kind for H is given in § 48.

By combining this result with those of the preceding article, we see that

$$P_{2n}-P_{2n-2}-P_{2n-4}+P_{2n-10}+P_{2n-14}-\&c.,$$

$$P_n+2P_{n-2}+2P_{n-8}+2P_{n-18}+2P_{n-28}+\&c.,$$

$$Q(2n)-3Q(2n-1)+5Q(2n-3)-7Q(2n-6)+\&c.$$

are all three equal; and that

$$P_m-P_{m-2}-P_{m-4}+P_{m-10}+P_{m-14}-\&c.,$$

$$-2P_n-2P_{n-4}-2P_{n-12}-2P_{n-24}-2P_{n-40}-\&c.,$$

$$Q(m)-3Q(m-1)+5Q(m-3)-7Q(m-6)+\&c.,$$

where $m=2n+1$, are all three equal.

42. From the equation

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1+q+q^3+q^6+q^{10}+\&c.)$$
$$=(1+P_1q^2+P_2q^4+P_3q^6+\&c.)(1-q-q^3+q^6+q^7-\&c.)$$

we may deduce the formulæ

$$P_{2n}+P_{2n-1}+P_{2n-3}+P_{2n-6}+\&c.$$
$$=P_n-P_{n-1}-P_{n-6}+P_{n-11}+P_{n-13}-P_{n-20}-\&c.$$

and
$$P_m+P_{m-1}+P_{m-3}+P_{m-6}+\&c.$$
$$=-P_n+P_{n-2}+P_{n-3}-P_{n-7}-P_{n-17}+P_{n-25}+\&c.,$$

where $n=2n+1$.

The numbers 1, 6, 11, ... which occur in the first formula are the halves of the even pentagonal numbers. The corresponding numbers 2, 3, 7, ... in the second formula are the halves of the uneven pentagonal numbers diminished by unity.

Similarly, from the equation

$$\{1+Q(1)q+Q(2)q^2+Q(3)q^3+\&c.\}(1-3q^2+5q^6-7q^{12}+9q^{20}-\&c.)$$
$$=(1+P_1q^2+P_2q^4+P_3q^6+\&c.)(1+q+q^3+q^6+q^{10}+\&c.),$$

we find

$$Q(2n)-3Q(2n-2)+5Q(2n-6)-7Q(2n-12)+\&c.$$
$$=P_n+P_{n-3}+P_{n-5}+P_{n-14}+P_{n-18}+P_{n-33}+\&c.$$

and $Q(m)-3Q(m-2)+5Q(m-6)-7Q(m-12)+\&c.$

$$= P_n+P_{n-1}+P_{n-7}+P_{n-10}+P_{n-22}+P_{n-27}+\&c.,$$

where $m = 2n+1$.

The numbers 3, 5, 14, ... which occur in the first formula are the halves of the even triangular numbers. The corresponding numbers 1, 7, 10, ... in the second formula are the halves of the uneven triangular numbers diminished by unity.

43. It can be shown that

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)\{1+P(1)\,q^3+P(2)\,q^4+P(3)\,q^6+\&c.\}$$
$$= 1-2q+2q^4-2q^9+2q^{16}-\&c.,$$

and

$$(1+P_1q^2+P_2q^4+P_3q^6+\&c.)\{1+P(1)\,q+P(2)\,q^3+P(3)\,q^5+\&c.\}$$
$$= 1+q+q^3+q^6+q^{10}+\&c.$$

By equating the coefficients of q^n in these equations, we find that

$$P_n+P(1)\,P_{n-2}+P(2)\,P_{n-4}+P(3)\,P_{n-6}+\&c. = 0,\ +2,\ \text{or}\ -2,$$

according as n is not a square, is an even square, or is an uneven square; and that

$$P(n)+P_1P(n-2)+P_2P(n-4)+P_3P(n-6)+\&c. = 0 \text{ or } 1,$$

according as n is not, or is, a triangular number.*

44. We have also the formula

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1+P_1q^2+P_2q^4+P_3q^6+\&c.)$$
$$= 1+\chi(5)\,q+\chi(9)\,q^2+\chi(13)\,q^3+\&c.,$$

where $\chi(n)$ is the function considered in § 19.

By equating coefficients, we find

$$P_n+P_1P_{n-2}+P_2P_{n-4}+P_3P_{n-6}+\&c. = \chi(4n+1).$$

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* The value of the expression

$$P_n+P(1)\,P_{n-1}+P(2)\,P_{n-2}+\ldots+P(n-1)\,P_1+P$$

was given in § 36.

Elliptic Function Expressions for Series involving the Function G, §§ 45–47.

45. Denoting $\frac{2K}{\pi}$ by ρ, we have, in Elliptic Functions, the following formulæ giving the values of the series which have been principally considered in this paper :—

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}} = 2^{\frac{1}{2}}q^{\frac{1}{12}}(1-q^2-q^4+q^{10}+q^{14}-\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2^{\frac{1}{2}}q^{\frac{1}{4}}(1+P_1q^2+P_2q^4+P_3q^6+\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{5}{2}} = 2q^{\frac{1}{4}}(1-3q^2+5q^6-7q^{12}+9q^{20}-\&c.).$$

These formulæ may be also written

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}} = 2^{\frac{1}{2}}(q^{\frac{1}{12}}-q^{\frac{25}{12}}-q^{\frac{49}{12}}+q^{\frac{121}{12}}+q^{\frac{169}{12}}-\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}(q^{\frac{1}{4}}+P_1q^{\frac{3}{4}}+P_2q^{\frac{5}{4}}+P_3q^{\frac{7}{4}}+\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2(q^{\frac{1}{4}}-3q^{\frac{3}{4}}+5q^{\frac{5}{4}}-7q^{\frac{7}{4}}+9q^{\frac{9}{4}}-\&c.);$$

or, by expressing only the general term of the series and using $G(12n+1)$ in place of P_n in the second formula,

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{12}m^2},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}\Sigma_0^{\infty}G(12n+1)\,q^{\frac{1}{4}(12n+1)},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{\frac{1}{4}m^2},$$

where n represents any number, and m any uneven number. The coefficients $\left(\frac{3}{m}\right)$ and $\left(\frac{-1}{m}\right)$ are Legendre's symbols (as extended by Jacobi) expressing the quadratic character of 3 and of -1 with respect to m, so that $\left(\frac{-1}{m}\right) = (-1)^{\frac{1}{2}(m-1)}$.

46. By substituting in this group of formulæ q^3, q^5 and $-q^5$ for q, we obtain also the following groups :—

$$k^{\frac{1}{2}}k'^{\frac{1}{12}}\rho^{\frac{1}{2}} = 2^{\frac{1}{2}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{12}m^2},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}\Sigma_0^{\infty}G(12n+1)\,q^{\frac{1}{4}(12n+1)},$$

$$kk'^{\frac{1}{2}}\rho^{\frac{1}{2}} \;\; = 4\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{\frac{1}{4}m^2};$$

$$k^{\frac{1}{12}}k'^{\frac{1}{4}}\rho^{\frac{1}{3}} = 2^{\frac{1}{3}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{12}m^2},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}\Sigma_0^{\infty}\, G(12n+1)\, q^{\frac{1}{12}(12n+1)},$$

$$k^{\frac{1}{4}}k'\,\rho^{\frac{3}{4}} \;\; = 2^{\frac{1}{3}}\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{\frac{1}{4}m^2};$$

$$k^{\frac{1}{12}}k'^{\frac{1}{12}}\rho^{\frac{1}{3}} = 2^{\frac{1}{3}}\Sigma_1^{\infty}\left(\frac{6}{m}\right)q^{\frac{1}{12}m^2},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \;\; = 2^{\frac{1}{2}}\Sigma_0^{\infty}\,(-1)^n\, G(12n+1)\, q^{\frac{1}{12}(12n+1)},$$

$$k^{\frac{1}{4}}k'^{\frac{1}{4}}\rho^{\frac{3}{4}} \;\; = 2^{\frac{1}{3}}\Sigma_1^{\infty}\left(\frac{-2}{m}\right)mq^{\frac{1}{4}m^2}.$$

The symbol $\left(\frac{-2}{m}\right)$ is equal to $(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}$. The symbols $\left(\frac{3}{m}\right)$ and $\left(\frac{6}{m}\right)$ are supposed to be zero whenever m is not prime to 3 or 6 respectively.

It may be remarked that $\left(\frac{3}{m}\right) = (-1)^{\frac{1}{2}(m\pm1)}$, where in the exponent that sign is to be taken which makes $m\pm1$ divisible by 6; m being supposed not to be a multiple of 3, as in that case $\left(\frac{3}{m}\right)$ is zero.

Thus, also, $\left(\frac{6}{m}\right) = \left(\frac{3}{m}\right)\left(\frac{2}{m}\right) = (-1)^{\frac{1}{2}(m\pm1)+\frac{1}{8}(m^2-1)}$.

47. The following formulæ may also be added:—

$$H\,(\tfrac{2}{3}K) = 3^{\frac{1}{4}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{4}m^2},$$

$$H^3(\tfrac{2}{3}K) = 3\,\Sigma_0^{\infty}\, G(12n+1)q^{\frac{1}{4}(12n+1)},$$

$$H^3(\tfrac{2}{3}K) = 3^{\frac{3}{4}}\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{\frac{1}{4}m^2};$$

where the function H is the same as in the *Fundamenta Nova.*

Linear Relations involving $H(n)$, and connecting $H(n)$ and $E(n)$, §§ 48, 49.

48. We may obtain various formulæ, of the kind considered in §§ 33–44, relating to the function $H(n)$ which was defined in § 28

as the excess of the number of divisors of n which $\equiv 1$, mod. 3, over those which $\equiv 2$, mod. 3. We obtain also, by the same methods, relations involving both $H(n)$ and $E(n)$.

From § 28,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = 2\Sigma_0^{\infty} H(4n+1)\, q^{4n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{4n^2} \times \Sigma_{-\infty}^{\infty} q^{3m^2} = 2\Sigma_0^{\infty} H(4n+3)\, q^{4n+3},$$

where n and m denote, as before, any number and any uneven number respectively.

From these formulæ we may deduce

$$\frac{H(1)+H(9)\,q^3+H(17)\,q^4+\&c.}{H(5)\,q+H(13)\,q^3+H(21)\,q^5+\&c.} = \frac{1+2q^{12}+2q^{48}+2q^{108}+\&c.}{2q^3+2q^{27}+2q^{75}+\&c.},$$

$$\frac{H(3)+H(11)\,q^2+H(19)\,q^4+\&c.}{H(7)\,q+H(15)\,q^3+H(23)\,q^5+\&c.} = \frac{1+2q^4+2q^{16}+2q^{36}+\&c.}{2q+2q^9+2q^{25}+\&c.},$$

whence, by equating coefficients, we find that, if $n \equiv 5$, mod. 8,

$$H(n)-2H(n-12)+2H(n-48)-2H(n-108)+\&c. = 0,$$

and that, if $n \equiv 7$, mod. 8,

$$H(n)-2H(n-4)+2H(n-16)-2H(n-36)+\&c. = 0.$$

49. From § 20,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{4n^2} = 2\Sigma_0^{\infty} E(4n+1)\, q^{4n+1},$$

$$\left\{\Sigma_{-\infty}^{\infty} q^{m^2}\right\}^2 = 4\Sigma_0^{\infty} E(4n+1)\, q^{8n+2},$$

$$\left\{\Sigma_{-\infty}^{\infty} q^{4n^2}\right\}^2 = 1+4\Sigma_1^{\infty} E(n)\, q^{4n};$$

and by combining these identities with those involving H in the preceding article, and equating coefficients, we may obtain the four following formulæ :—

(i.) If $n \equiv 1$, mod. 4,

$$H(n)+2H(n-4)+2H(n-16)+2H(n-36)+\&c.$$
$$= E(n)+2E(n-12)+2E(n-48)+2E(n-108)+\&c.$$

In the first line the numbers 4, 16, 36, ... are the even squares; the corresponding numbers in the second line are the triples of these squares.

(ii.) If $$p = 4n+1, \quad s = 8n+1,$$

where n is any number, then

$$H(s)+H(s-8)+H(s-24)+H(s-48)+\&c.$$
$$= E(p)+2E(p-24)+2E(p-96)+2E(p-216)+\&c.$$

The numbers 8, 24, 48, ... are the uneven squares diminished by unity. The numbers 24, 96, 216, ... are the squares multiplied by 24.

(iii.) If $$l = 4n-3, \quad t = 8n+5,$$

where n is any number, then

$$H(t)+H(t-8)+H(t-24)+H(t-48)+\&c.$$
$$= 2E(l)+2E(l-48)+2E(l-144)+2E(l-288)+\&c.$$

The numbers 8, 24, 48, ... are as above. The numbers 48, 144, 288 are these numbers multiplied by 6.

(iv.) If $$r = 4n+3,$$

$$H(r)+2H(r-4)+2H(r-16)+2H(r-36)+\&c.$$
$$= 4E(n)+4E(n-6)+4E(n-18)+4E(n-36)+\&c.$$

The numbers 6, 18, 36, ... are the triangular numbers multiplied by 6. The quantity $E(0)$, when it occurs, is to have the value $\frac{1}{4}$.

It may be remarked that we find also that, if $p = 4n+1$,

$$E(p)+2E(p-4)+2E(p-16)+2E(p-36)+\&c.$$
$$= 4E(n)+4E(n-2)+4E(n-6)+4E(n-12)+\&c.,$$

where, as in (iv.), $E(0)$ is to have the value $\frac{1}{4}$.

The Functions G and H, §§ 50–56.

50. Taking the second pair of formulæ in § 31, and changing the sign of q^{12} in the second of them, we have

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} H(12n+1)\, q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} (-1)^n G(12n+1)\, q^{12n+1}.$$

These equations express the theorems:—

(i.) If a^2 denote an even square, and b^2 an uneven square not divisible by 3, then the number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+b^2$ is equal to $4H(p)$.

(ii.) If a^2 denote an even square, c^2 a square of the form $(12n+1)^2$, and d^2 a square of the form $(12n+5)^2$, then the excess of the number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+c^2$ over the number of representations by the form $3a^2+d^2$ is equal to $4(-1)^{\frac{1}{12}(p-1)}G(p)$.

Combining these two theorems, we see that (a^2, b^2, c^2 having the same meanings as above) the number of representations of p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+c^2$ is equal to

$$2\{H(p)+(-1)^{\frac{1}{12}(p-1)}G(p)\},$$

and by the form $3a^2+d^2$ is equal to

$$2\{H(p)-(-1)^{\frac{1}{12}(p-1)}G(p)\}.$$

51. From the formulæ in the preceding section, it follows that

$$\frac{\Sigma_{-\infty}^{\infty}q^{(6n+1)^2}}{\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2}}=\frac{\Sigma_0^{\infty}H(12n+1)q^{12n+1}}{\Sigma_0^{\infty}(-1)^n G(12n+1)q^{12n+1}},$$

whence

$$\frac{1+H(13)q+H(25)q^3+H(37)q^5+\&c.}{1-G(13)q+G(25)q^3-G(37)q^5+\&c.}=\frac{1+q^3+q^4+q^{10}+q^{14}+\&c.}{1-q^3-q^4+q^{10}+q^{14}-\&c.}.$$

Equating coefficients, we find that, if $n\equiv 1$, mod. 12,

$$G(n)+G(n-24)+G(n-48)+G(n-120)+G(n-168)+\&c.$$
$$=(-1)^{\frac{1}{12}(n-1)}\{H(n)-H(n-24)-H(n-48)+H(n-120)$$
$$+H(n-168)-\&c.)\}.$$

The numbers 24, 48, 120, ... are the uneven squares which are not divisible by 3 diminished by unity.

52. It can be shown that

$$1+P_1q^3+P_2q^4+P_3q^6+\&c.$$
$$=(1-q-q^3+q^5+q^7-\&c.)(1+q+q^3+q^6+q^{12}+\&c.),$$

whence we deduce

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2}\times\Sigma_{-\infty}^{\infty}q^{3m^2}=2\Sigma_0^{\infty}G(12n+1)q^{48n+4},$$

where m denotes any uneven number.

This equation expresses the theorem:

If m^2 denote an uneven square, c^2 a square of the form $(12n+1)^2$, and d^2 a square of the form $(12n+5)^2$, then the excess of the number

of representations of any number p (necessarily $\equiv 2$, mod. 24) by the form $3m^2+c^2$ over the number of representations by the form $3m^2+d^2$ is equal to zero, if $p \equiv 2$, mod. 48, and is equal to $8G(\frac{1}{4}p)$ if $p \equiv 4$, mod. 48.

53. From the formulæ,

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2} \times \Sigma_1^{\infty} q^{3m^2} = \Sigma_0^{\infty} G(12n+1)\, q^{48n+4},$$

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty}(-1)^n G(12n+1)\, q^{12n+1},$$

we obtain, by division,

$$\frac{1+P_1q^4+P_2q^8+P_3q^{12}+\&c.}{1-P_1q+P_2q^3-P_3q^8+\&c.} = \frac{1+q^2+q^6+q^{12}+q^{20}+\&c.}{1+2q+2q^4+2q^9+2q^{16}+\&c.};$$

whence, by equating coefficients, we find that

$$P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.$$

is equal to zero, if $n \equiv 2$ or 3, mod. 4; is equal to

$$P_u+2P_{u-1}+2P_{u-4}+2P_{u-9}+\&c.,$$

if $n \equiv 0$, mod. 4, where $u = \frac{1}{4}n$; and is equal to

$$-2P_v-2P_{v-2}-2P_{v-6}-2P_{v-12}-\&c.,$$

if $n \equiv 1$, mod. 4, where $v = \frac{1}{4}(n-1)$.

54. The second of the above results shows that, for all values of n,

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+\&c. = P_{4n}+P_{4n-2}+P_{4n-6}+P_{4n-12}+\&c.$$

Now, in § 39 it was shown that

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+\&c.$$

is equal to zero for all uneven values of n. It follows, therefore, that

$$P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.$$

is equal to zero whenever n is the quadruple of an uneven number, that is to say, whenever $n \equiv 4$, mod. 8.

The third result shows that, if $p = 4n+1$, then, for all values of n,

$$P_p+P_{p-2}+P_{p-6}+P_{p-12}+\&c. = -2\{P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.\}.$$

Similarly, if $r = 16n+5$,

$$P_r+P_{r-2}+P_{r-6}+P_{r-12}+\&c. = 4\{P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.\};$$

and in general, if

$$\nu = 4^r n + \tfrac{1}{3}(4^r - 1),$$

$$P_\nu + P_{\nu-2} + P_{\nu-4} + P_{\nu-12} + \&c. = (-2)^r \{P_n + P_{n-2} + P_{n-4} + P_{n-12} + \&c.\}.$$

As a particular case, putting $n = 0$, we see that, if $\nu = \frac{1}{3}(4^r - 1)$,

$$P_\nu + P_{\nu-2} + P_{\nu-4} + P_{\nu-12} + \&c. = (-2)^r.$$

55. It has been shown in the two preceding articles that

$$P_n + P_{n-2} + P_{n-4} + P_{n-12} + \&c.$$

is equal to zero, if $n \equiv 2$ or 3, mod. 4, or $\equiv 4$, mod. 8; and it follows, therefore, from the theorem just proved, that it is also equal to zero, if

$$n = 4^r c + \tfrac{1}{3}(4^r - 1),$$

where c is any number $\equiv 2$ or 3, mod. 4, or $\equiv 4$, mod. 8; or, which is the same thing, $\equiv 2$, 3, 4, 6, or 7, mod. 8. For example, putting $r = 1$, and $c \equiv 2$ or 3, mod. 4, we see that it is equal to zero when $n \equiv 9$ or 13, mod. 16.

We have found, therefore, that the expression

$$P_n + P_{n-2} + P_{n-4} + P_{n-12} + \&c.$$

is equal to zero, if $n \equiv 2$, 3, 4, 6, or 7, mod. 8; or if $n \equiv 2$, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14, or 15, mod. 16; and so on. In other words, if the expression does not vanish, n must necessarily $\equiv 0$, 1, or 5, mod. 8; or $\equiv 0$, 1, 5, 8, mod. 16; and so on.

56. The class of functions to which $G(n)$ belongs possesses two distinct kinds of properties, both of which are available for their calculation. The one kind, depending upon the divisors of n, is practically contained in the theorem

$$\phi(pq) = \phi(p)\,\phi(q),$$

where p and q are relatively prime; the other consists of the various recurring formulæ (such as those considered in §§ 33–44) in which $\phi(n)$ is expressed as a finite series of ϕ's of arguments less than n, and separated from n by numbers of special forms, such as squares, pentagonal numbers, &c. Properties of the former kind depend, as it were, on the arithmetical structure of the function $\phi(n)$, and cannot, so far as I see, be directly derived from the series in which the functions $\phi(n)$ appear as coefficients. On the other hand, the recurring formulæ

connecting the values of $\phi(n)$ are readily obtained by equating coefficients; but the kind of consideration by which such a theorem as

$$\phi(pq) = \phi(p)\,\phi(q)$$

is proved, appears to afford no clue to the discovery of these formulæ, nor to their demonstration when found. Amongst functions of this class [*i.e.*, which possess the property $\phi(pq) = \phi(p)\,\phi(q)$, and satisfy also various recurring formulæ] may be mentioned the functions E, χ, H, σ, referred to in this paper, and ζ, λ (*Proc. Lond. Math. Soc.*, Vol. xv., p. 109, and *Quart. Journ.*, Vol. xx., p. 145).

Complex Multiplication Moduli of Elliptic Functions for the Determinants -53 *and* -61. *By* Professor G. B. Mathews.

[*Read Dec. 12th*, 1889.]

The following note contains the solution of two cases of complex multiplication referred to by Mr. Greenhill in his paper on the subject (*Proceedings*, Vol. xix.) as being hitherto unsolved: viz., those in which $\Delta = 53, 61$, respectively.

As my results are merely supplementary to the paper just quoted, and the method of procedure (which is essentially Hermite's) is there sufficiently explained and illustrated (see, for instance, pp. 326–328), I have not thought it necessary to do more than give the actual algebraical work.

Applying Hermite's method to the modular equation for $n = 31$, the values of

$$2n - \rho^2 \quad (\rho = 1, 3, 5, 7)$$

are $\qquad 61, \quad 53, \quad 37, \quad 13.$

We have to put $\qquad \kappa\lambda = w^4, \quad \kappa'\lambda' = 2iw^2,$

and then in Russell's notation

$$P = w + \sqrt{(1+i)\,w} + 1,$$

$$Q = w\sqrt{(1+i)\,w} + \sqrt{(1+i)\,w} + w,$$

$$R = w\sqrt{(1+i)\,w}.$$

Hence $$P^2 = w^3+(3+i)\,w+1+2\,(w+1)\,\sqrt{(1+i)\,w},$$

$$P^2-4Q = w^3+(-1+i)\,w+1-2\,(w+1)\,\sqrt{(1+i)\,w},$$

$$PR = (1+i)\,w^3+w\,(w+1)\,\sqrt{(1+i)\,w}.$$

Thus the modular equation

$$(P^2-4Q)^3-4PR = 0$$

becomes $$w^6+(2+6i)\,w^5+(6+2i)\,w^3+(2+6i)\,w+1$$
$$-4\,(w+1)(w^3+iw+1)\,\sqrt{(1+i)\,w} = 0.$$

Transposing and squaring, we get a reciprocal equation in w, viz.,

$$w^8-4\,(3+i)\,w^7-4\,(5+9i)\,w^6+4\,(9-i)\,w^5+2\,(1-12i)\,w^4+\ldots = 0.$$

Putting $$w+\frac{1}{w} = y,$$

we get $y^4-4\,(3+i)\,y^3-12\,(2+3i)\,y^2+8\,(9+i)\,y+4\,(11+12i) = 0,$

or $(y+3+i)\,\{y^3-5\,(3+i)\,y^2+2\,(8-3i)\,y+2\,(9+5i)\} = 0;$

where $$y+3+i = 0$$

corresponds to $$\Delta = 13.$$

Taking the other part, and putting

$$x = y^2-2 = \sqrt{\kappa\lambda}+\frac{1}{\sqrt{\kappa\lambda}},$$

$$x^3-162\,(1+i)\,x^2-360ix+432\,(1-i) = 0.$$

Let $$x = (1+i)\,\sqrt{\beta};$$

then $2i\,(1+i)\,\beta\sqrt{\beta}-2i162\,(1+i)\,\beta-360i\,(1+i)\,\sqrt{\beta}+432\,(1-i) = 0,$

$$\beta^3-162\beta-180\beta^2-216 = 0,$$

or $$\sqrt{\beta} = 6m,$$

where $$m^3-27m^2-5m-1 = 0.$$

The discriminant of this cubic is $2^{10}\,.\,61$;

Hermite's $$a = 144m^4.$$

Now, if $$y = m^2,$$

$$y\,(y-5)^2-(27y+1)^2 = 0,$$

or $$y^8-739y^4-29y-1=0;$$

and if $$a=y^3,$$

$$a(a-29)^3-(739a+1)^2=0,$$

or, finally, $$a^3-546179a^2-637a-1=0,$$

where the $a=\frac{1}{144}$ of Hermite's a.

The other two cases, $\Delta=37$ and $\Delta=53$, are solved by writing

$$\sqrt{-(1+i)w} \quad \text{instead of} \quad \sqrt{(1+i)w}.$$

The equation in w is

$$w^8-4(1-i)w^7+4(19+15i)w^6-4(37+39i)w^5$$
$$+10(13+36i)w^4+\ldots=0;$$

and the equation in y,

$$y^4-4(1-i)y^3+12(6+5i)y^2-8(17+21i)y-20(1-12i)=0,$$

that is,

$$(y+3-7i)\{y^3-(7-11i)y^2+2(8-11i)y-10(3-i)\}=0.$$

Here $$y+3-7i=0$$

corresponds to $\Delta=37$; the other part gives, on putting as before

$$y^3-2=x,$$

the equation $x^3+110(1+i)x^2+536ix-1232(1-i)=0.$

Hence $$\beta^4+110\beta+268\beta^2+616=0,$$

or $$\sqrt{\beta}=2m,$$

where $$m^3+55m^2+67m+77=0.$$

As a verification, the discriminant of this equation is found to be

$$2^{10}.5^4.53.$$

Finally, m having been calculated, Hermite's a is given by

$$a=2^8.m^4.$$

Thursday, February 13th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

Concerning Semi-Invariants: Mr. S. Roberts.

Ether-Squirts: Prof. K. Pearson.

On Class-Invariants: Prof. G. B. Mathews.

Note on the Imaginary Roots of an Equation: Prof. Cayley.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVI., No. 284.

"Educational Times" for February.

"The Elements of Astronomy, with a Uranography," by Dr. C. A. Young; 8vo; Boston and London, 1890.

"The Lightning Calculator," by C. H. B. de Botwor; 8vo pamphlet; London.

"Proceedings of the Canadian Institute," No. 152 (whole Vol. XXV.); Toronto.

"Bulletin des Sciences Mathématiques," Tome XIV., Jan. 1890.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 7–10; Roma, 1889.

"Journal für die reine und angewandte Mathematik," Band CVI., Heft 1.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 12; Band XIV., Stück 1.

"Memorias de la Sociedad Cientifica 'Antonio Alzate,'" Tomo II., No. 12; Tomo III., Nos. 1 and 2; Mexico.

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"Annali di Matematica," Tomo XVII°, Fasc. 3°.

"Rendiconti del Circolo Matematico di Palermo," Tomo III., Fasc. 6; Nov., Dic., 1889.

"Tidsskrift for Mathematik," Række V., Aargang 6; Hefte 1–7; Kjobenhavn, 1888.

"Nyt Tidsskrift for Mathematik," Aargang 1; Nr. 1, A; Aargang 1, Nr. 1, B.

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G. Lejeune Dirichlet's Werke—Erster Band; 4to (Edited by L. Kronecker); Berlin, 1889.

"Journal of the College of Science," Imperial University, Japan; Vol. III., Part 3; Tokyo.

Concerning Semi-Invariants. *By* SAMUEL ROBERTS, F.R.S.

[*Read Feb.* 13*th*, 1890.]

1. I remarked in a former paper (*Quarterly Journal of Mathematics*, Vol. VII., p. 17, 1865), that if we expand the product

$$\left\{a_0+a_1(x-y)+a_2\frac{(x-y)^2}{1.2}+a_3\frac{(x-y)^3}{1.2.3}+\ldots\right\}$$
$$\times\left\{a_0+a_1(y-z)+a_2\frac{(y-z)^2}{1.2}+a_3\frac{(y-z)^3}{1.2.3}+\ldots\right\}$$
$$\times \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\times\left\{a_0+a_1(u-v)+a_2\frac{(u-v)^2}{1.2}+a_3\frac{(u-v)^3}{1.2.3}+\ldots\right\}$$
$$\times\left\{a_0+a_1(v-x)+a_2\frac{(v-x)^2}{1.2}+a_3\frac{(v-x)^3}{1.2.3}+\ldots\right\},$$

the coefficients of powers and products of powers of x, y, z, ... u, v are the leading coefficients of certain covariants of quantics, to a numerical factor *près*. These quantics are of the form

$$(a_0, a_1, a_2 \ldots a_p)(x, y)^p;$$

i.e., the literal coefficients are affected with binomial multipliers.

In the place cited, a proof of the general theorem is not explicitly given. The line of demonstration is, however, indicated in the case of the product of two indefinitely extended series,

$$\left(a_0+a_1x+\frac{a_2x^2}{1.2}+\frac{a_3x^3}{1.2.3}+\ldots\right)\left(a_0-a_1x+\frac{a_2x^2}{1.2}-\frac{a_3x^3}{1.2.3}+\ldots\right)\ldots.(a).$$

If Π_1, Π_2 denote the two operators

$$\left(a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+a_3\frac{d}{da_2}+\ldots\right)$$

and

$$\left(a_1'\frac{d}{da_0'}+a_2'\frac{d}{da_1'}+a_3'\frac{d}{da_2'}+\ldots\right),$$

then $(\Pi_1-\Pi_2)^p\, a_0 a_0'$ is a semi-invariant* annulled by the operator

$$\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\ldots\right)+\left(a_0'\frac{d}{da_1'}+2a_1'\frac{d}{da_2'}+3a_2'\frac{d}{da_3'}+\ldots\right).$$

* When a "semi-invariant" is mentioned, it is supposed to belong to binary forms with binomial coefficients. The literal coefficients, as a_0, a_1, a_2, &c., are often styled "elements."

The product (a) is symbolically expressed by

$$e^{x(\Pi_1-\Pi_2)}\, a_0 a_0',$$

the accents being removed after operation. It is implied that the general result was obtained by means of the same artifice from the semi-invariant

$$\left[(x-y)\,\Pi_1+(y-z)\,\Pi_2+\ldots+(u-v)\,\Pi_{k-1}+(v-x)\,\Pi_k\right]^p a_0 a_0' \ldots a^{(k-1)r} \quad \ldots\ldots\ldots\ldots(b),$$

where Π_i operates on a set of a's with i accents, and corresponds in form to Π_1. It is apparent, by taking a limited form, as

$$(x-y)\,\Pi_1+(y-z)\,\Pi_2+(z-x)\,\Pi_3,$$

for the general one, that (b) is annulled by the operator

$$\Sigma\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\ldots\right),$$

the summation extending to all the sets of elements.

Mr. Hammond has informed me that a result closely connected with (b) has been given by Mr. Kempe. In fact, since (b) may be written

$$\left[x\,(\Pi_1-\Pi_k)+y\,(\Pi_2-\Pi_1)+\ldots \right.$$
$$\left.\ldots+u\,(\Pi_{k-1}-\Pi_{k-2})+v\,(\Pi_k-\Pi_{k-1})\right]^p a_0 a_0' \ldots a_0^{(k-1)r},$$

and $x, y, z, \ldots u, v$ are indeterminate, it is plain that

$$(\Pi_1-\Pi_k)^m\,(\Pi_2-\Pi_1)^n \ldots (\Pi_k-\Pi_{k-1})^t\, a_0 a_0' \ldots a_0^{(k-1)r}$$

is a semi-invariant, and this is Mr. Kempe's form. It is a particular case of a more general one. For we may write for (b)

$$(X_1\Pi_1+X_2\Pi_2+\ldots+X_k\Pi_k)^p\, a_0 a_0' \ldots a_0^{(k-1)r},$$

subject to $$X_1+X_2+\ldots+X_k=0.$$

Now, let $(a_{11}, a_{12}, \ldots a_{1k})$, $(a_{21}, a_{22}, \ldots a_{2k})$, $\ldots\ldots$ $(a_{k1}, a_{k2}, \ldots a_{kk})$

be k sets of numbers such that

$$\sum_{p=k}^{p=1} a_{1p} = \sum_{p=k}^{p=1} a_{2p} = \ldots = \sum_{p=k}^{p=1} a_{kp} = 0\,;$$

then $Y_1(a_{11}\Pi_1+a_{12}\Pi_2+\ldots+a_{1k}\Pi_k)+Y_2(a_{21}\Pi_1+a_{22}\Pi_2+\ldots+a_{2k}\Pi_k)+\ldots$

$$\ldots+Y_k(a_{k1}\Pi_1+a_{k2}\Pi_2+\ldots+a_{kk}\Pi_k)$$

$$=(Y_1a_{11}+Y_2a_{21}+\ldots+Y_ka_{k1})\Pi_1+(Y_1a_{12}+Y_2a_{22}+\ldots+Y_ka_{k2})\Pi_2+\ldots$$

$$\ldots+(Y_1a_{1k}+Y_2a_{2k}+\ldots+Y_ka_{kk})\Pi_k,$$

and the sum of the coefficients of Π_1, Π_2, ... Π_k in the last expression is zero by hypothesis. Consequently,

$$(a_{11}\Pi_1+a_{12}\Pi_2+\ldots+a_{1k}\Pi_k)^m(a_{21}\Pi_1+a_{22}\Pi_2+\ldots+a_{2k}\Pi_k)^n\ldots$$

$$\ldots(a_{k1}\Pi_1+a_{k2}\Pi_2+\ldots+a_{kk}\Pi_k)^s a_0a_0'\ldots a_0^{(k-1)r}$$

is a semi-invariant.

With reference again to the theorem first before given, it is, on the face of it, a very particular case, but I did not dwell on it at the time. I propose now to investigate some general theorems which it suggests. It will be unnecessary in the present connection to repeat the phrase "to a numerical factor *près*."

Let u, u_1, u_2, &c. represent the indefinitely extended function

$$a_0+a_1yX+\frac{a_2y^2X^2}{1.2}+\frac{a_3y^3X^3}{1.2.3}+\ldots$$

(which I hereafter call "elementary"), and its successive derived functions with respect to X. Also, let Φ denote the semi-invariant annihilator

$$a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\ldots+ia_{i-1}\frac{d}{da_i}+\ldots,$$

continued indefinitely. We have immediately, and by differentiation with respect to X, since Φ and $\frac{d}{dX}$ are commutative,

$$\left.\begin{aligned}\Phi u &= yXu\\ \Phi u_1 &= yXu_1+yu\\ \Phi u_2 &= yXu_2+2yu_1\\ \ldots\quad &\quad\ldots\quad\quad\ldots\quad\quad\ldots\\ \Phi u_m &= yXu_m+myu_{m-1}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

2. Consider F, an integral homogeneous function of any number k of elementary sub-functions of the type of u, and of limited series of their successive derived functions relative to X, beginning in each

case with the first. We may represent such a group of functions of yX by $(u, u_1, u_2 \ldots)_k$, the literal coefficients of powers of yX being different in each set corresponding to u, u_1, u_2 &c. The operator Φ must now be of the extended form

$$\Sigma\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\ldots\right)_k,$$

the sign of summation covering the whole group of elements, and F, of the degree μ in the elementary sub-functions, will stand for

$$F\left[(u, u_1, u_2 \ldots)_k\right].$$

It must always be borne in mind that the sub-functions are indefinitely continued.

By means of (A), we obtain

$$\begin{aligned}\Phi F &= \Sigma\left(\Phi u\frac{d}{du}+\Phi u_1\frac{d}{du_1}+\Phi u_2\frac{d}{du_2}+\ldots\right)_k F\\ &= yX\Sigma\left(u\frac{d}{du}+u_1\frac{d}{du_1}+u_2\frac{d}{du_2}+\ldots\right)_k F\\ &\quad +y\Sigma\left(u\frac{d}{du_1}+2u_1\frac{d}{du_2}+3u_2\frac{d}{du_3}+\ldots\right)_k F.\end{aligned}$$

If, therefore, F is of the form of a semi-invariant as to the u's considered as elements, and is annulled by the operator

$$\Sigma\left(u\frac{d}{du_1}+2u_1\frac{d}{du_2}+3u_2\frac{d}{du_3}+\ldots\right)_k,$$

or, what is the same thing, if the leading term of F is isobaric, and a semi-invariant annulled by Φ, the result is, immediately and by differentiation with respect to X,

$$\left.\begin{aligned}\Phi F &= y\mu XF,\\ \Phi F_1 &= y\mu XF_1+y\mu F,\\ \Phi F_2 &= y\mu XF_2+2y\mu F_1,\\ &\ldots \quad \ldots \quad \ldots \quad \ldots\\ \Phi F_m &= y\mu XF_m+my\mu F_{m-1},\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

where F_1, F_2, &c. are the successive derived functions of F with respect to X.

Let the development of F, according to ascending powers of X, be

$$A_0+A_1X+\frac{A_2X^2}{1.2}+\ldots+\frac{A_rX^r}{r!}+\ldots,$$

and let Π denote the operator

$$\Sigma\left(a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+a_3\frac{d}{da_2}+\ldots\right)_k;$$

then $$A_r = y^r\Pi^r A_0,$$

and the function F may be compendiously represented by

$$e^{yX\Pi}A_0.^*$$

3. Next, suppose $\mathbf{F}$ to be an integral homogeneous function of any number l of sub-functions of the type of F, and of limited series of their successive derived functions (beginning with the first) with respect to X, but depending on different groups of elementary sub-functions.

Also, each sub-function of the type F may have a different μ and a different y. The operator Φ is to be extended to all the literal coefficients of the elementary sub-functions, or, as we may say, to all the elements.

Then we have, analogously,

$$\begin{aligned}\Phi\mathbf{F} &= \Sigma\left(\Phi F\frac{d}{dF}+\Phi F_1\frac{d}{dF_1}+\Phi_2F_2\frac{d}{dF_2}+\ldots\right)_l\mathbf{F}\\ &= X\Sigma\left[y\mu\left(F\frac{d}{dF}+F_1\frac{d}{dF_1}+F_2\frac{d}{dF_2}+\ldots\right)_l\right]\mathbf{F}\\ &\quad+\Sigma\left[y\mu\left(F\frac{d}{dF}+2F_1\frac{d}{dF_2}+3F_2\frac{d}{dF_3}+\ldots\right)_l\right]\mathbf{F}.\end{aligned}$$

If in this expression we make throughout the summations $y\mu = 1$, &c., and assume that $\mathbf{F}$ is a semi-invariant as to $(F, F_1, \&c.)_l$, considered as elements, and is annulled by the operator

$$\Sigma\left(F\frac{d}{dF_1}+2F_1\frac{d}{dF_2}+3F_2\frac{d}{dF_3}+\ldots\right)_l,$$

we have immediately, and by differentiation with respect to X,

$$\left.\begin{aligned}\Phi\mathbf{F} &= \pi X\mathbf{F},\\ \Phi\mathbf{F}_1 &= \pi X\mathbf{F}_1+\pi\mathbf{F},\\ \Phi\mathbf{F}_2 &= \pi X\mathbf{F}_2+2\pi\mathbf{F}_1,\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\\ \Phi\mathbf{F}_m &= \pi X\mathbf{F}_m+m\pi\mathbf{F}_{m-1},\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C),$$

* *Quarterly Journal of Mathematics*, Vol. IV., "On Arbogast's Calculus of Derivations," p. 195; communicated July 1860, printed February 1861.

where π is the degree of $\mathbf{F}$ in the sub-functions of the type F, and their derived functions with respect to X, and $\mathbf{F}_1$, $\mathbf{F}_2$, &c., $\mathbf{F}_n$ are the successive derived functions of $\mathbf{F}$ with respect to X.

We might proceed to treat in a similar manner, and with similar result, a function of sub-functions of the type of $\mathbf{F}$, and of limited series of their successive derived functions with respect to X, and so on indefinitely.

4. The relation which a function of the type $\mathbf{F}$ bears to one of the type F, when $y = 1$, may be exhibited in the following manner. Since, as we have seen, an elementary sub-function of the type u is represented, when $y = 1$, by $e^{X\Pi} a_0$, and a sub-function of the type F is represented, when $y = \mu^{-1}$, by $e^{X\Pi/\mu} A_0$, where A_0 is a semi-invariant relative to k groups of a-elements, and is annulled by Φ, we may write, for a function F,

$$\psi\left[(e^{X\Pi} a_0,\ e^{X\Pi} a_1,\ e^{X\Pi} a_2,\ \ldots)_k\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (c);$$

and for a function $\mathbf{F}$,

$$\psi\left[\left(e^{X\Pi/\mu} A_0,\ e^{X\Pi/\mu} \frac{A_1}{\mu},\ e^{X\Pi/\mu} \frac{A_2}{\mu^2},\ \ldots\right)_k\right] \quad \ldots\ldots\ldots\ldots (d);$$

that is to say, if in (c) we change a_0 into A_0, a_1 into $\frac{A_1}{\mu}$, a_2 into $\frac{A_2}{\mu^2}$, and so forth, we obtain (d).

Further, let $\mathbf{F}$ become $\mathbf{F}_0$ when X is made equal to zero. From the first equation of (C) we get

$$\Phi \mathbf{F}_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (D),$$

and $\mathbf{F}_0$ stands for

$$\mathbf{F}\left[\left(A_0,\ \frac{\Pi A_0}{\mu},\ \frac{\Pi^2 A_0}{\mu^2}\ \ldots\ \frac{\Pi^i A_0}{\mu^i},\ \ldots\right)_i\right],$$

the corresponding substitutions being made for all the leading terms of the sub-functions of the type F, and for those of their successive derived functions. But in the condition

$$\Sigma\left(F \frac{d}{dF_1} + 2F_1 \frac{d}{dF_2} + 3F_2 \frac{d}{dF_3} + \ldots\right)_i \mathbf{F} = 0,$$

we may write any letters b, b_1, b_2, b_3, &c. for F, F_1, F_2, F_3, &c.; for these F's are here regarded in the light of elements of a binary form. The result (D) may therefore be interpreted as follows:—If in any semi-invariant of any system of independent binary forms involving

the elements b_0, b_1, b_2, &c., b_0', b_1', b_2', &c., and so forth, we substitute, for b_0, b_1, b_2, &c., A_0, $\frac{\Pi A_0}{\mu}$, $\frac{\Pi^2 A_0}{\mu^2}$, &c., and for b_0', b_1', b_2', &c., A_0', $\frac{\Pi A_0'}{\mu_1}$, $\frac{\Pi^2 A_0'}{\mu_1^2}$, &c., and so forth, A_0, A_0', &c. being semi-invariants of the degrees μ, μ_1, &c. relative to the elements (represented by a's) of any systems of independent binary forms, and Π being the operator

$$\Sigma\left(a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+a_3\frac{d}{da_2}+\ldots\right),$$

where the summation extends to all the systems of elements; the result will be a semi-invariant of all the substituted systems of forms.*

As a simple example, take the semi-invariant $b_0b_2'-2b_1b_1'+b_2b_0'$, and substitute therein (a_0, a_1, a_2) for (b_0, b_1, b_2); and

$$\left(a_0a_2-a_1^2,\ \frac{a_0a_3-a_1a_2}{2},\ \frac{a_0a_4-a_2^2}{4}\right) \text{ for } (b_0', b_1', b_2').$$

We have as the result,

$$\tfrac{1}{4}a_0(a_0a_4-4a_1a_3+3a_2^2),$$

which, multiplied by 4, is the source of the covariant found by the product of a quintic, and the S of its quartic emanant; or, considering a_0 as the source of a quartic, the other factor is the S of the quartic. Of course, a semi-invariant becomes an invariant if the order of the covariant is zero.

5. I proceed now to theorems, directly analogous to the one quoted at the beginning of § 1.

Take any number l of functions $v, w, \ldots z$ of the pattern of u, only writing $X_1, X_2, \ldots X_l$ for X therein, and making $y=1$. Then, by (A),

$$\Phi u.v.w\ldots z=(X+X_1+\ldots+X_l)\,u.v.w\ldots z=0,$$

if

$$X+X_1+\ldots+X_l=0.$$

This is the theorem quoted in § 1, since we may write X for $x-y$, X_1 for $y-z$, ... X_l for $v-x$.

We obtain a more general result by applying similar considerations to (B), putting $y=1$ and xX for X. We have, subject to the conditions prescribed,

$$\Phi F=x\mu XF.$$

* This is equivalent to "Théorème V." enunciated by M. Perrin (*Comptes Rendus*, 1887, pp. 1258–1260).

Take any number l of functions $U, V, W, \ldots Z$ of the pattern of F, and involving elementary sub-functions of xX, x_1X, x_2X, &c., with different coefficients, and limited series of their successive derived functions (beginning in each case with the first) with respect to xX, x_1X, x_2X, &c.). Let the degrees of homogeneity of U, V, W, &c. Z be, respectively, $\mu, \mu_1, \mu_2, \ldots \mu_{l-1}$; then

$$\Phi U.\ V.\ W \ldots Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(E)},$$

provided that $$x\mu + x_1\mu_1 + \ldots + x_{l-1}\mu_{l-1} = 0,$$

and Φ be adapted to all the systems of elementary sub-functions involved.

Since $$\Phi U^a = aU^{a-1}\Phi U,$$

we may write for (E),

$$\Phi U^a.\ V^{a_1}.\ W^{a_2} \ldots Z^{a_{l-1}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(F)},$$

provided that $x\mu a + x_1\mu_1 a_1 + x_2\mu_2 a_2 + \ldots + x_{l-1}\mu_{l-1}a_{l-1} = 0.$

This is really only another rendering of (E); because, if U is of the degree of homogeneity μ, U^a will be of the degree of homogeneity $a\mu$. But in the last form we can have $x, x_1, \ldots x_{l-1}$, $a, a_1, a_2, \ldots a_{l-1}$ negative or fractional.

When such is the case, however, we have still to develop according to ascending powers of X, and the coefficients of the expansion will consequently be affected with denominators consisting of powers of the leading terms of the negative powers expanded, and, if the powers expanded are fractional, with factors consisting of corresponding fractional powers of the leading terms. In the case of negative indices, the numerators of the coefficients, after reduction to a common denominator, will be integral semi-invariants. In the case of fractional indices, the irrational factor or factors may be removed. An easy example will, perhaps, make this clearer than description in words.

Consider the product of the indefinitely extended series

$$\left(a_0 + a_1xX + a_2\frac{x^2X^2}{1.2} + \ldots\right)^a \left(b_0 + b_1x_1X + b_2\frac{x_1^2X^2}{1.2} + \ldots\right)^\beta,$$

where $b_0 = a_0a_2 - a_1^2$, $b_1 = a_0a_3 - a_1a_2$, $b_2 = a_0a_4 - a_2^2$, &c.

Here, since b_0 is of the second degree in the a's, we must have

$$ax + 2\beta x_1 = 0,$$

or, say, $$x = 2\beta, \quad x_1 = -\alpha.$$

The coefficient of X in the product is

$$a_0^{\alpha} b_0^{\beta} \left\{ 2\alpha\beta \frac{a_1}{a_0} - \alpha\beta \frac{b_0}{b_0} \right\},$$

or $$-\alpha\beta a_0^{\alpha-1} (a_0 a_2 - a_1^2)^{\beta-1} \{a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3\}.$$

The coefficient of X^2 is

$$a_0^{\alpha} b_0^{\beta} \left\{ -2\alpha^2\beta^2 \frac{a_1 b_1}{a_0 b_0} + \frac{4\alpha\beta^2}{1.2} \frac{a_2}{a_0} + \frac{\alpha^2\beta}{1.2} \frac{b_2}{b_0} + \frac{\beta(\beta-1)}{1.2} \alpha^2 \frac{b_1^2}{b_0^2} + \frac{\alpha(\alpha-1)}{1.2} 4\beta^2 \frac{a_1^2}{a_0^2} \right\},$$

giving

$$\alpha\beta \frac{a_0^{\alpha-2} (a_0 a_2 - a_1^2)^{\beta-2}}{2} \{\alpha\beta (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3)^2 + 4\beta (a_0 a_2 - a_1^2)^3 + \alpha a_0^3 [a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3]\}.$$

In these two cases, then, we can remove the factors with indices depending on α and β. The remaining factors, when, if necessary, a proper numerical factor has been applied, become rational and integer semi-invariants.

6. To illustrate (E), let $w_0, w_0', w_0'', \ldots w_0^{(k)'}$ be semi-invariants of any systems of binary forms whose elements are a_0, a_1, a_2, &c.; a_0', a_1', a_2', &c.; and so on.

Let Π, as before, be the developing operator

$$\Sigma \left(a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \ldots \right).$$

The equation (E) shows that

$$e^{x_0 X \Pi} w_0, \quad e^{x_1 X \Pi} w_0', \quad e^{x_2 X \Pi} w_0'' \ldots e^{x_k X \Pi} w_0^{(k)'} \quad \ldots\ldots\ldots (G)$$

is a semi-invariant, if

$$\mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_k x_k = 0,$$

where $\mu_0, \mu_1, \ldots \mu_k$ are the degrees of $w_0, w_0', \ldots w^{(k)'}$, respectively, in the elements.

If the general operator Π be broken up into portions, viz., Π_0 operating exclusively on w_0, Π_1 operating on w_0' exclusively, and

generally Π_p operating on $w^{(p)'}$ exclusively, the coefficient of $\frac{X^m}{m!}$, viz.,

$$(x_0'\Pi_0+x_1\Pi_1+x_2\Pi_2+\ldots+x_k\Pi_k)^m\, w_0w_0'\ldots w_0^{(k)'},$$

is a semi-invariant.

If $k=1$, we may put $x_0=\mu_1$, $x_1=-\mu_0$,*

and the coefficient becomes

$$(\mu_1\Pi_0-\mu_0\Pi_1)^m\, w_0w'.$$

This is the extension of M. D'Ocagne's theorem given by M. Perrin (*Comptes Rendus*, 1887, pp. 1097–1099, 1258–1260; see also "Développements sur la théorie des formes binaires," par M. J. Deruyts, *Bull. de l'Académie royale de Belgique*, 3me série, t. xiv., p. 65).

If $w_0, w_0', \ldots w_0^{(k)'}$ be identically the same, and w_i be written for $\Pi^i w_0$, the coefficient of X^m is

$$w_0^k\frac{w_m}{m!}\Sigma x_0^m+w_0^{k-1}w_1\frac{w_{m-1}}{m-1!}\Sigma x_0x^{m-1}+w_0^{k-1}\frac{w_2}{2!}\frac{w_{m-2}}{m-2!}\Sigma x_0^2x_1^{m-2}$$

$$+w_0^{k-2}w_1^2\frac{w_{m-2}}{m-2!}\Sigma x_0x_1x_2^{m-1}+\&c.,$$

each term containing a symmetric function of $x_0x_1\ldots x_k$ as a factor. If we denote $\Sigma x_0^\alpha x_1^\beta\ldots x_r^\rho$, where $\alpha, \beta, \ldots \rho$ are different integers, by $[\alpha\beta\gamma\ldots\rho]$, and if any, as α, β, γ, be equal, by $[\alpha^3\ldots\rho]$, the coefficient may be symbolically expressed by

$$\Sigma\left[\left(\frac{w_\alpha}{\alpha!}\right)^p\left(\frac{w_\beta}{\beta!}\right)^q\left(\frac{w_\gamma}{\gamma!}\right)^r\ldots\right]\left[\alpha^p\beta^q\gamma^r\ldots\right],$$

with $\qquad \alpha p+\beta q+\gamma r+\ldots=m, \quad p+q+r+\ldots=k+1,$

and one of the numbers $\alpha, \beta, \gamma, \ldots$ may be zero, and $0!=1$, as usual in such cases.

Since $\qquad x_0+x_1+\ldots+x_k=0,$

---

* Like the expression (c) in § 1,

$$\left(\frac{a_{00}}{\mu}\Pi_0+\frac{a_{01}}{\mu_1}\Pi_1+\ldots+\frac{a_{0k}}{\mu_k}\Pi_k\right)^m\left(\frac{a_{10}}{\mu}\Pi_0+\frac{a_{11}}{\mu_1}\Pi_1+\ldots+\frac{a_{1k}}{\mu_k}\Pi_k\right)^n\ldots$$

$$\ldots\left(\frac{a_{k0}}{\mu}\Pi_0+\frac{a_{k1}}{\mu_1}\Pi_1+\ldots+\frac{a_{kk}}{\mu_k}\Pi_k\right)^s w_0w_0'\ldots w^{(k)},$$

is a semi-invariant, if $\sum_{p=k}^{p=0}a_{0p}=\sum_{p=k}^{p=0}a_{1p}=\ldots=\sum_{p=k}^{p=0}a_{kp}=0.$

we may substitute for the symmetric functions their values in terms of the coefficients of

$$y^{k+1}+K_1y^{k-1}-K_2y^{k-2}+\ldots+(-)^{k+1}K_{k+1}=0,$$

having as roots $x_0, x_1, \ldots x_k$.

If, still supposing $w_0, w_0', \ldots w_0^{(k)'}$ identically the same, we divide each factor of the product (G) by w_0, and put generally

$$\left(1+\frac{w_1}{w_0}x_pX+\frac{w_2}{w_0}\frac{x_p^2X^2}{1.2}+\ldots\right)=(1-\lambda x_pX)(1-\lambda' x_pX)(1-\lambda'' x_pX)\ldots,$$

we may select, from each of the expressions corresponding to the factors of (G), the factor corresponding to $1-\lambda x_pX$.

Multiplying these factors together, we get

$$1+L_1\lambda^1X^1-L_2\lambda^3X^3+L_4\lambda^4X^4-\&c.+(-)^{k+1}L_{k+1}\lambda^{k+1}X^{k+1},$$

and (G) becomes, after division by w_0^{k+1}, an infinite product of factors of the above form. The coefficients of the powers of X in the expansion will therefore be non-unitary symmetric functions of the roots $\lambda, \lambda', \lambda''$, &c.

As a very particular case of (G), we may take the function

$$\left(a_0+a_1X+\frac{a_2X^2}{1.2}+\ldots\right)^k\left(a_0-a_1kX+a_2\frac{k^2X^2}{1.2}-\ldots\right).$$

The coefficient of X^m here is

$$a_0A_m-a_1kA_{m-1}+a_2\frac{k^2}{1.2}A_{m-2}-\ldots+(-)^m a_mA_0,$$

where $A_m, A_{m-1}, \ldots$ are the coefficients of $X^m, X^{m-1}, \ldots$ in the expansion of the k^{th} power. Since this is also a case of (F), we are at liberty to make k negative or fractional, in which case the coefficients will be affected with powers of a_0.

If $w_0, w_0', \ldots w_0^{(k)'}$ are semi-invariants as to one set of elements a_0, a_1, a_2, &c., it follows that

$$e^{X\Pi}\frac{w_0w_0'\ldots w_0^{(k)'}}{a^{(\rho_0+\rho_1+\ldots+\rho_k)}}$$

is a semi-invariant, and the coefficients of the different powers of X are so. Here $w_0, w_0', \ldots w^{(k)'}$ are not assumed to be of the same degree

in the elements. This, however, is not really more general than

$$e^{X\Pi}\frac{w_0}{a_0^{\pi}},$$

since the product of semi-invariants is a semi-invariant.

7. Let us now take any number l of functions $\mathbf{U}, \mathbf{V}, \mathbf{W}, \ldots \mathbf{Z}$ of the type $\mathbf{F}$, and let their degrees in the F functions, and in limited series of their derived functions with respect to X, be respectively $\pi_0, \pi_1, \ldots \pi_{l-1}$. Also, write for X, x_0X in $\mathbf{U}$, x_1X in $\mathbf{V}$, and so on, and let Φ be adapted to all the elementary sub-functions involved; then, by means of (C), we obtain

$$\Phi\,\mathbf{U}^{\alpha_0}.\mathbf{V}^{\alpha_1}.\mathbf{W}^{\alpha_2}\ldots\mathbf{Z}^{\alpha_{l-1}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(H)},$$

subject to $x_0a_0\pi_0 + x_1a_1\pi_1 + x_2a_2\pi_2 + \ldots + x_{l-1}a_{l-1}\pi_{l-1} = 0.$

In this result, as in the analogous case (F), we may make $x_0, x_1, \ldots x_{l-1}$, $a_0, a_1, \ldots a_{l-1}$ negative or fractional, subject to considerations similar to those mentioned in regard to (F). When the product affected by Φ is expanded according to powers of X, the coefficients of those powers are semi-invariants relative to the systems of elements involved.

It seems unnecessary to write down more general expressions analogous to (F) and (H), for the last is already so general that the simplest examples are laborious.

8. There is a special case to be noted, which does not quite fall in with the general results.

Let U be a function of the type F, and of the form

$$A_0 + A_1X + \frac{A_2X^2}{1.2} + \ldots \text{ ad inf.},$$

where A_0 is a semi-invariant of any system of binary forms, and

$$A_r = \Pi^r A_0.$$

Then
$$\Phi \log U = \frac{\Phi U}{U} = \mu X \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(K)},$$

μ being the degree of U in the elements; and from this it appears that the coefficients of powers of X higher than the first in the expansion of $\log U$ are semi-invariants. Similarly,

$$\Phi\,\frac{U_1}{U} = \mu,$$

showing that the coefficients of the powers of X, after that of X^0 (or the constant term), in the expansion of $\frac{U_1}{U}$, are semi-invariants. But this is, of course, a direct inference from (K).

Since $$\log U = \log A_0 + \log\left(1 + \frac{A_1}{A_0}X + \frac{A_2}{A_0}\frac{X^2}{1.2} + \ldots\right),$$

it follows that the sums of the powers of the reciprocal roots of $U = 0$ (higher than the first) are semi-invariants, and Major Macmahon's inference that any integral symmetric function of the reciprocal of the roots of

$$1 + bx + \frac{cx^2}{1.2} + \ldots = 0,$$

in which the indices of the roots are greater than unity, is a semi-invariant, holds for $U = 0$, subject to a power of A_0 appearing as a denominator. That is to say, such symmetrical functions of the reciprocal roots of $U = 0$ are semi-invariants not only with regard to A_0, A_1, A_2, &c., considered as elements, but with regard to the elements involved in the A's.

9. If, instead of an elementary function U, we take

$$v = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots,$$

indefinitely continued, the coefficients being deprived of the exponential multipliers, the coefficient of x^{m+1} in v^{-m} is a semi-invariant, relative to the elements a_0, a_1, a_2, &c.

For, operating with

$$\Phi = a_0\frac{d}{da_1} + 2a_1\frac{d}{da_2} + 3a_2\frac{d}{da_3} + \ldots$$

on v^{-m}, we get

$$-m(a_0x + 2a_1x^2 + 3a_2x^3 + \ldots)v^{-m-1} = -mxv^{-m} + x^2\frac{d}{dx}v^{-m},$$

and the coefficient of x^{m+1} in this is

$$(\text{co. of } x^m) \text{ in } -mv^{-m} + (\text{co. of } x^{m-1}) \text{ in } \frac{d}{dx}v^{-m} = 0,$$

so that Φ annuls the coefficient of x^{m+1} in v^{-m}.

Now, if we revert the series (indefinitely continued)

$$z = a_0x + a_1x^2 + a_2x^3 + \ldots,$$

we have, putting Ω for $a_1\frac{d}{da_0}+2a_2\frac{d}{da_1}+3a_3\frac{d}{da_2}+\ldots$,

$$x=a_0^{-1}z+\frac{1}{2}\Omega a_0^{-2}.z^2+\frac{1}{3}\frac{\Omega^2a_0^{-3}}{1.2}z^3+\frac{1}{4}\frac{\Omega^3a_0^{-4}}{1.2.3}z^4+\ldots,$$

$$x^{-1}=a_0z^{-1}+\frac{a_1}{a_0}-\frac{\Omega^2a_0^{-1}}{1.2}z-\frac{1}{2}\frac{\Omega^3a_0^{-2}}{1.2.3}z^2-\frac{1}{3}\frac{\Omega^4a_0^{-3}}{1.2.3.4}z^3-\ldots.*$$

But $\frac{\Omega^{m+1}a_0^{-m}}{m+1!}$ is the coefficient of x^{m+1} in v^{-m}, and therefore is a semi-invariant as to the elements a_0, a_1, a_2, &c. Hence in the value of x^{-1} in terms of z, and the elements a_0, a_1, a_2, &c., every term after the first two is a semi-invariant multiplied by a power z.

Of course, the value of $\frac{1}{x}$ in question is the negative first power of the familiar series

$$\frac{z}{a_0}-\frac{a_1}{a_0^2}z^2+\frac{2a_1^2-a_0a_2}{a_0^5}z^3-\frac{a_0^2a_3-5a_0a_1a_2+5a_1^3}{a_0^7}z^4+\ldots.$$

If we divide this by z/a_0, and call the quotient S, the expansion of S^{-1} according to ascending powers of x has for the coefficient of x^p the sum of the homogeneous products with repetitions (of the p^{th} order) of the reciprocals of the roots of $S=0$. Therefore, when a_0 is put equal to unity, and p being greater than unity, the sum of such homogeneous products is a semi-invariant relative to the elements 1, a_1, a_2, &c. There is some analogy here to the known property of

$$1+a_1x+\frac{a_2x^2}{1.2}+\&c.=0.$$

The foregoing result may be extended by taking, instead of the function v, a function

$$V=A_0+A_1x+A_2x^2+A_3x^3+\&c.,$$

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* West's Mathematical Treatises, edited by Leslie, 1838, pp. 161–3. We have to notice that in West's notation, if O be

$$\beta\frac{d}{d\alpha}+2\gamma\frac{d}{d\beta}+3\delta\frac{d}{d\gamma}+\ldots,$$

$$\left(\frac{a^p}{1}\right)',\ \left(\frac{a^p}{1}\right)'',\ \left(\frac{a^p}{1}\right)'''\ldots \text{ mean } Oa^p,\ \frac{O^2a^p}{1.2},\ \frac{O^3a^p}{1.2.3}\ldots\ldots.$$

Cf. Quart. Journal of Math., Vol. IV., p. 198.

where A_0 is a semi-invariant of the degree μ with elements of any system of binary forms,

$$\Pi = \Sigma\left[\left(a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+a_3\frac{d}{da_2}+\ldots\right)\right],$$

the summation extending to all the elements, and

$$A_p = \Pi^p A_0.$$

In fact, by the comparison of coefficients on each side of the first equation in (B), we have

$$\Phi A_p = \mu p A_{p-1},^*$$

and
$$\begin{aligned}\Phi V^{-m} &= -m\,(\Phi A_0+\Phi A_1 . x+\Phi A_2 . x^2+\ldots)\,V^{-m-1}\\ &= -m\mu\,(A_0x+2A_1x^2+3A_2x^3+\ldots)\,V^{-m-1}\\ &= -m\mu x V^{-m}+\mu x^2\frac{d}{dx}V^{-m}.\end{aligned}$$

The coefficient of x^{m+1} in this is zero. We can now revert the series

$$Z = A_0x+A_1x^2+A_2x^3+\ldots,$$

and reason as in the simpler case, with analogous result. In like manner, we might proceed to take, instead of the function V of the type F, a function of the type $\mathbf{F}$, and so on.

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* This was given and employed by M. Perrin (1887). His notes on the subject of semi-invariants appear to have been suggested by a particular theorem due to M. D'Ocagne, viz., if w_p, w_q be two semi-invariants of the binary form

$$a_0x^n+\binom{n}{1}a_1x^{n-1}+\ldots+a_ny^n,$$

and if p and q be their respective orders, and Δ is the operator

$$a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+\ldots+a_n\frac{d}{da_{n-1}},$$

then $qw_q\Delta w_p - pw_p\Delta_q$ is a semi-invariant.

On Class-Invariants. *By* Professor G. B. Mathews.

[*Read Feb.* 13*th*, 1890.]

The modular function usually denoted by $J(\omega)$ is of such fundamental importance, especially in connexion with the arithmetical aspect of the modular-function theory, that it seems desirable to discuss at least some cases of complex multiplication with direct reference to the (J, J') modular equations. This is what I have endeavoured to do in the following pages in the cases enumerated by Klein (*Math. Annalen*, t. xiv.*), where the equation defining J' or $J(\omega/n)$ as a function of $J(\omega)$ is of "Geschlecht" zero; that is to say, when $n = 2, 3, 4, 5, 7, 13$, respectively. The notation of the paper just quoted has been adopted, so far as possible, throughout; in some respects it would perhaps be more convenient to express the results in terms of $j(\omega) = 1728\, J(\omega)$; but there is no difficulty in making the alteration if desired. Since the interest of this paper is, I think, principally algebraical, only so much of the general theory of modular functions has been prefixed as seems necessary to show the connexion of the results with the arithmetical theory of definite quadratic forms.†

If ω is any complex quantity with a positive imaginary part, then, putting $e^{\pi i\omega} = q$, the function $J(\omega)$ is defined by

$$J(\omega) = \frac{\left(\frac{1}{12} + 20\Sigma m^3 \frac{q^{2m}}{1-q^{2m}}\right)^3}{q^2\Pi(1-q^{2m})^{24}}. \qquad (m = 1, 2, 3 \ldots \infty.)$$

The fundamental property of J is that, ω', ω being different complex quantities,

$$J(\omega') = J(\omega)$$

if, and only if,

$$\omega' = \frac{a\omega + \beta}{\gamma\omega + \delta},$$

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* A summary of this paper will be found in *Proc. Lond. Math. Soc.*, Vol. ix., p. 123.

† Perhaps the most elementary introduction to the theory of modular functions is Hurwitz's memoir, *Math. Ann.*, xviii. (1881), p. 528, where further references will be found.

where α, β, γ, δ are real integers such that

$$\alpha\delta-\beta\gamma=1.$$

If, on the other hand, $\alpha\delta-\beta\gamma=n$,

an integer different from unity, then $J(\omega')$ is an algebraical function of $J(\omega)$, and the equation connecting $J(\omega')$ and $J(\omega)$, or say J' and J, may be called the (J, J') modular equation of the n^{th} order.

Now, suppose ω is a root (with positive imaginary part) of a primitive quadratic form (a, b, c), so that

$$a\omega^2+2b\omega+c=0;$$

then, according as the form is properly or improperly primitive, this gives

$$a\omega+2b=-\frac{c}{\omega},$$

or

$$\tfrac{1}{2}a\omega+b=-\frac{\frac{1}{2}c}{\omega};$$

from which we infer $J(a\omega)=J(c\omega)$,

or $J(\tfrac{1}{2}a\omega)=J(\tfrac{1}{2}c\omega)$,

respectively.

In particular, the forms $(1, b, n)$, $(2, b, 2n)$ lead to the relation

$$J(\omega)=J(n\omega);$$

so that, if in the (J, J') modular equation of the n^{th} order we put $J'=J$, one of the roots of the resulting equation will be a class-invariant for the form considered; and the values of J thus obtained will all belong to forms of determinants $-\Delta$ for which Δ is positive, and of the form $n-b^2$ or $4n-b^2$, respectively. It will be seen, then, that the method is substantially identical with that of Hermite, in his "Théorie des Équations Modulaires," the difference being that the irrational quantities $\sqrt[4]{\kappa}$, $\sqrt[4]{\kappa'}$ do not appear; instead of them, we have the quantity which Klein calls τ, an algebraic function of J in terms of which (for the special values of n here considered) all other modular functions can be rationally expressed (*cf.* Klein, *loc. cit.*, p. 127).

In further explanation of the following results it may be added that, for $n=2, 3, 4, 5, 7, 13$, Klein expresses J in the form $\phi(\tau)/\psi(\tau)$, where $\phi(\tau)$, $\psi(\tau)$ are integral functions of τ, and J', that is, $J(\omega/n)$, is the same function of τ', where τ and τ' are connected by a lineo-linear

equation of the form

$$\tau\tau' = C \quad \text{or} \quad \tau+\tau' = C,$$

where C is a constant. By putting $J' = J$, and eliminating τ', an equation in τ is obtained by solving which the singular values of τ, and thence of J, are found. In addition to the values of τ and J, I have determined in almost all cases the classes of quadratic forms to which they belong.

From the point of view of the complex multiplication of elliptic functions,

$$J = \frac{4}{27} \cdot \frac{(1-\kappa^2\kappa'^2)^3}{\kappa^4\kappa'^4},$$

where κ, κ' are Legendre's moduli; and ω, the root of any one of the corresponding quadratic forms, is the ratio of two elementary periods of the elliptic functions connected with these particular moduli. Thus, for instance, $J(1, 0, n)$ is connected with that case of complex multiplication for which $K'/K = \sqrt{n}$. The arithmetical, or modular function theory, seems, however, more fundamental, and the results have therefore been exhibited in their connexion therewith.

I. $n = 2$.

Here $$27J = \frac{(4\tau-1)^3}{\tau} = \frac{(4-\tau)^3}{\tau^2},$$

therefore $$\tau(4\tau-1)^3 - (4-\tau)^3 = 0,$$

or $$(\tau^2-1)(64\tau^2-47\tau+64) = 0;^*$$

and the values of τ and J are as follows:—

$$\tau = 1, \qquad J = 1,$$

corresponding to the class $(1, 0, 1) \sim (1, 1, 2)$;

$$\tau = -1, \qquad J = \frac{5^3}{3^3},$$

corresponding to $(1, 0, 2)$.

As a verification, we know that in the latter case

$$\kappa = \sqrt{2}-1, \quad \kappa'/\kappa = \sqrt{2};$$

whence $$\kappa^2 = (\sqrt{2}-1)^2, \quad \kappa'^2 = 2(\sqrt{2}-1),$$

$$1-\kappa^2\kappa'^2 = 5(3-2\sqrt{2}) = 5(\sqrt{2}-1)^2,$$

* Here and elsewhere the sign $\sim$ denotes equivalence.

and $$J = \frac{4}{27}\frac{(1-\kappa^2\kappa'^2)^3}{\kappa^4\kappa'^4} = \frac{5^3}{3^3}.$$

The other values of r are

$$r = \frac{47 \pm 45i\sqrt{7}}{128},$$

giving $$4r-1 = \frac{15\,(1 \pm 3i\sqrt{7})}{32},$$

$$J = \frac{15^3 . 128\,(1 \pm 3i\sqrt{7})^3}{27 . 32^3\,(47 \pm 45i\sqrt{7})}$$

$$= -\frac{15^3 . 128 . 4}{27 . 32^3} = -\frac{5^3}{2^6},$$

and the corresponding class is (2, 1, 4).

II. $n = 3$.

$$-64J = \frac{(r-1)(9r-1)^3}{r} = \frac{(1-r)(9-r)^3}{r^3},$$

$$(r-1)\left\{r^3(9r-1)^3-(r-9)^3\right\} = 0,$$

or $$(r-1)(r+1)\left\{729\left(r+\frac{1}{r}\right)^2-972\left(r+\frac{1}{r}\right)-460\right\} = 0;$$

hence

$$r = 1, \qquad J = 0,$$

$$r = -1, \qquad J = \frac{5^3}{2^2},$$

$$r = \frac{23 \pm 10i\sqrt{2}}{27}, \qquad J = \frac{5^3}{3^3},$$

$$r = \frac{-5 \pm 8i\sqrt{11}}{27}, \qquad J = -\frac{2^9}{3^3}.$$

The corresponding classes are

$$(2, 3, 6) \sim (2, 1, 2),\ (1, 0, 3), \quad (1, 1, 3) \sim (1, 0, 2),\ (2, 1, 6).$$

(See also Greenhill, *Proc. Lond. Math. Soc.*, Vol. XIX., pp. 306, 351.)

III. $n = 4$.

$$108J = \frac{(16-16r+r^2)^3}{r^4(1-r)} = \frac{(1+14r+r^2)^3}{r\,(1-r)^4},$$

$$r\,(r-1)\left\{(r-1)^3\,(r^2-16r+16)^3+r^3\,(r^2+14r+1)^3\right\} = 0.$$

$r = 0$ or 1 gives $J = \infty$, which has to be rejected. Taking the other factor, we have either

$$\text{(i.)} \quad (r-1)(r^3-16r+16)+r(r^3+14r+1) = 0,$$

that is,
$$(2r-1)(r^3-r+16) = 0;$$

or $$\text{(ii.)} \quad r^6-3r^5+756r^4-1507r^3+1761r^2-1008r+256 = 0,$$

that is,
$$(r^3-r+1)\{(r^3-r)^2+752(r^3-r)+256\} = 0.$$

Solving for r, we get

$$r = \frac{1}{2}, \qquad J = \frac{11^3}{2^8},$$

$$r = \frac{1 \pm 3i\sqrt{7}}{2}, \qquad J = -\frac{5^3}{2^6},$$

$$r = \frac{1 \pm i\sqrt{3}}{2}, \qquad J = \frac{5^3}{2^2},$$

$$r = \frac{1 \pm i\sqrt{3}\,(16+7\sqrt{5})}{2},$$

$$J = \frac{5\sqrt{5}}{2^{10}}(\sqrt{5}-2)^3(4-\sqrt{5})^3(\sqrt{5}+1)(\sqrt{5}+3)^3$$

$$= \frac{5\sqrt{5}}{2^8}(4-\sqrt{5})^3(\sqrt{5}-1)^2,$$

$$r = \frac{1 \pm i\sqrt{3}\,(16-7\sqrt{5})}{2},$$

$$J = \frac{5\sqrt{5}}{2^{10}}(\sqrt{5}+2)^3(4+\sqrt{5})^3(\sqrt{5}-1)(\sqrt{5}-3)^3$$

$$= -\frac{5\sqrt{5}}{2^8}(4+\sqrt{5})^3(\sqrt{5}+1)^2.$$

The reduction in the last two cases is very curious. Putting

$$r^3-r+8(47+21\sqrt{5}) = 0,$$

one value of r is

$$r = \frac{1+i\sqrt{1503+672\sqrt{5}}}{2}$$

$$= \frac{1+i\sqrt{3}\,(16+7\sqrt{5})}{2}.$$

This gives

$$\frac{1+14\tau+\tau^2}{1-\tau} = \frac{3\,(16+7\sqrt{5})\,(-7\sqrt{5}+5i\sqrt{3})}{1-i\sqrt{3}\,(16+7\sqrt{5})}$$

$$= \frac{960\,(16+7\sqrt{5})}{(-1+i\sqrt{3})(112\sqrt{5}+240)}$$

$$= \frac{60\,(16+7\sqrt{5})}{(-1+i\sqrt{3})(7\sqrt{5}+15)}.$$

Now
$$16+7\sqrt{5} = (4-\sqrt{5})(9+4\sqrt{5}),$$
$$7+3\sqrt{5} = (3-\sqrt{5})(9+4\sqrt{5}),$$

and the expression reduces to

$$\frac{3\sqrt{5}\,(4-\sqrt{5})}{3-\sqrt{5}}(-1-i\sqrt{3}).$$

Also, $\tau\,(1-\tau) = 8\,(47+21\sqrt{5}) = 8\,(\sqrt{5}+2)^3(\sqrt{5}-1),$

and
$$3-\sqrt{5} = (\sqrt{5}-2)(\sqrt{5}+1);$$

therefore

$$108J = \frac{1}{8\,(\sqrt{5}-1)(\sqrt{5}+2)^3}\cdot\frac{27\cdot 5\sqrt{5}\,(4-\sqrt{5})^3}{(\sqrt{5}-2)^3\,(\sqrt{5}+1)^3}\cdot 8,$$

$$J = \frac{5\sqrt{5}\,(4-\sqrt{5})^3}{4^3\,(\sqrt{5}+1)^3}$$

$$= \frac{5\sqrt{5}}{2^8}(4-\sqrt{5})^3(\sqrt{5}-1)^2.$$

The cases $n = 7$ and $n = 13$ present some reductions of the same character, but even more complicated; I have not thought it worth while to give them explicitly.

The classes corresponding to the different values of J are

$$(1, 0, 4),\ (2, 1, 4),\ (1, 1, 4) \sim (1, 0, 3),\ (2, 1, 8),\ (4, 1, 4).$$

I have not determined which of the irrational values of J corresponds to (2, 1, 8), and which to (4, 1, 4).

IV. $n = 5$.

$$-1728J = \frac{(\tau^2-10\tau+5)^3}{\tau} = \frac{(\tau'^2-10\tau'+5)^3}{\tau'},$$

where
$$\tau\tau' = 125;$$

or, writing $5t$, $5t'$ for r, r', so that $tt' = 5$,

$$-\frac{1728J}{25} = \frac{(5t^3-10t+1)^3}{t} = \frac{(t^3-50t+125)^3}{5t^5},$$

whence

$$625t^{10}-3750t^9+7875t^8-6500t^7+1574t^6-7870t^4+162500t^3-984375t^2$$
$$+2343750t-1953125 = 0.$$

This is

$$(t^2-5)(625t^8-3750t^7+11000t^6-25250t^5$$
$$+56574t^4-126250t^3+275000t^2-468750t+390625) = 0.$$

Putting $t = y\sqrt{5}$ in the second factor, we get a reciprocal equation in y; and writing

$$u = \frac{5\sqrt{5}}{2}\left(y+\frac{1}{y}\right) = \frac{5}{2}\left(t+\frac{5}{t}\right),$$

the equation for u is

$$u^4-15u^3-15u^2+775u-1386 = 0,$$

that is,

$$(u-2)(u+7)(u-9)(u-11) = 0.$$

Hence the values of t are given by

$$t^2 - 5 = 0,$$
$$5t^2 - 4t+25 = 0,$$
$$5t^2+14t+25 = 0,$$
$$5t^2-18t+25 = 0,$$
$$5t^2-22t+25 = 0.$$

Calculating the values of J, we have—

$$t = \sqrt{5}, \qquad J = -\frac{5\sqrt{5}}{2^3.3^3}(\sqrt{5}-1)^3(\sqrt{5}-2)^3(\sqrt{5}+4)^3,$$

$$t = -\sqrt{5}, \qquad J = +\frac{5\sqrt{5}}{2^3.3^3}(\sqrt{5}+1)^3(\sqrt{5}+2)^3(4-\sqrt{5})^3,$$

$$t = \frac{2\pm 11i}{5}, \qquad J = \frac{11^3}{2^3},$$

$$t = \frac{-7\pm 2i\sqrt{19}}{5}, \qquad J = -2^9,$$

$$t = \frac{9\pm 2i\sqrt{11}}{5}, \qquad J = -\frac{2^9}{3^3},$$

$$t = \frac{11\pm 2i}{5}, \qquad J = 1.$$

As a verification of the second value of J, we have, for $\Delta = 5$,

$$\kappa\kappa' = \frac{\sqrt{5}-2}{2},$$

$$J = \frac{4}{27}\cdot\frac{(1-\kappa^2\kappa'^2)^3}{\kappa^4\kappa'^4}$$

$$= \frac{4}{27}\cdot\frac{\left\{1-\frac{9-4\sqrt{5}}{4}\right\}^3}{\frac{161-72\sqrt{5}}{16}}$$

$$= \frac{1}{27}\cdot\frac{(-5+4\sqrt{5})^3}{161-72\sqrt{5}}$$

$$= \frac{5\sqrt{5}(4-\sqrt{5})^3}{27}(161+72\sqrt{5})$$

$$= \frac{1}{8.27}\cdot 5\sqrt{5}(4-\sqrt{5})^3(\sqrt{5}+1)^3(\sqrt{5}+2)^3.$$

The corresponding classes for the last five values of J are

$$(1, 0, 5), \quad (1, 1, 5) \sim (1, 0, 4), \quad (2, 1, 10),$$
$$(2, 3, 10) \sim (2, 1, 6), \quad (1, 2, 5) \sim (1, 0, 1),$$

and for the first, presumably, (2, 1, 3).

V. $n = 7$.

Writing $7t$, $7t'$ for Klein's r, r',

$$1728J = \frac{(7t^3+13t+7)(49t^3+35t+1)^3}{t}$$

$$= \frac{(7t^3+13t+7)(t^3+35t+49)^3}{t^7}.$$

If $$7t^3+13t+7 = 0, \quad J = 0.$$

Next, put

$$t^6(49t^3+35t+1)^3 - (t^3+35t+49)^3 = 0;$$

that is,

$$(49t^4+35t^3-35t-49)$$
$$\times(2401t^8+3430t^7+1372t^6+1820t^5+3629t^4$$
$$+1820t^3+1372t^2+3430t+2401) = 0.$$

The first factor gives

$$(t^2-1)(49t^2+35t+49)=0;$$

putting $7\left(t+\frac{1}{t}\right)=y$ in the other,

$$y^4+10y^3-168y^2-1210y+5687=0,$$

or
$$(y-11)(y+11)(y^2+10y-47)=0,$$

and the remaining equations for t are

$$7t^2-11t+7=0,$$

$$7t^2+11t+7=0,$$

$$7t^2+(5+6\sqrt{2})\,t+7=0,$$

$$7t^2+(5-6\sqrt{2})\,t+7=0.$$

Hence we derive the following values for J:—

$t=1$, $\quad J=\frac{5^3.17^3}{2^6}$,

$t=-1$, $\quad J=-\frac{5^3}{2^6}$,

$t=\frac{-5\pm 3i\sqrt{19}}{14}$, $\quad J=-2^6$,

$t=\frac{11\pm 5i\sqrt{3}}{14}$, $\quad J=-\frac{2^6.5^3}{3^3}$,

$t=\frac{-11\pm 5i\sqrt{3}}{14}$, $\quad J=\frac{5^3}{2^3}$,

$t=\frac{-(5+6\sqrt{2})\pm i\sqrt{3}\,(5-2\sqrt{2})}{14}$, $\quad J=(\sqrt{2}-1)^3(5-2\sqrt{2})^3$,

$t=\frac{-(5-6\sqrt{2})\pm i\sqrt{3}\,(5+2\sqrt{2})}{14}$, $\quad J=(\sqrt{2}+1)^3(5+2\sqrt{2})^3$,

$t=\frac{-13\pm 3i\sqrt{3}}{14}$, $\quad J=0$.

The classes corresponding to the first five of these are

$$(1, 0, 7),\quad (2, 1, 4),\quad (2, 3, 14)\sim(2, 1, 10),$$

$$(2, 1, 14),\quad (1, 2, 7)\sim(1, 0, 3);$$

to the last belongs

$$(2, 5, 14) \sim (2, 1, 2).$$

Hence we conjecture that the remaining two correspond to (1, 0, 6) and (2, 0, 3); and this may be verified by means of Weber's results contained in *Math. Ann.*, XXXIII. (1889), p. 402. He finds that, for the principal class (1, 0, 6),

$$2\sqrt{\frac{\kappa'^2}{\kappa}} = 2\sqrt{2}(\sqrt{2}+1),$$

or

$$\frac{\kappa'^2}{\kappa} = (2+\sqrt{2})^2 = (6+4\sqrt{2}).$$

Now, if $\frac{\kappa'^2}{\kappa} = x$, it is easily seen that

$$J = \frac{4}{27} \cdot \frac{(1+x^2)^3}{x^2},$$

and this gives, in the present case,

$$\begin{aligned} J &= \frac{4}{27} \cdot \frac{(69+48\sqrt{2})^3}{(2+\sqrt{2})^4} = (23+16\sqrt{2})^3(\sqrt{2}-1) \\ &= \{(5+2\sqrt{2})(3+2\sqrt{2})\}^3(\sqrt{2}-1)^4 \\ &= (5+2\sqrt{2})^3(\sqrt{2}+1)^6(\sqrt{2}-1)^4 \\ &= (5+2\sqrt{2})^3(1+\sqrt{2})^2, \text{ as above.} \end{aligned}$$

VI. $n = 13$.

Putting $t\sqrt{13}$, $t'\sqrt{13}$ for Klein's r, r',

$$1728\sqrt{13} \cdot J$$

$$= \frac{(13t^2+5\sqrt{13} \cdot t+13)(169t^4+91\sqrt{13} \cdot t^3+260t^2+19\sqrt{13} \cdot t+1)^3}{t}$$

$$= \frac{(13t^2+5\sqrt{13} \cdot t+13)(t^4+19\sqrt{13} \cdot t^3+260t^2+91\sqrt{13} \cdot t+169)^3}{t^{13}}.$$

Hence the following alternative solutions:—

(i.) $13t^2+5\sqrt{13} \cdot t+13 = 0$, that is, $r^2+5r+13 = 0$;

(ii.) $169t^6+91\sqrt{13} \cdot t^5+260t^4+19\sqrt{13} \cdot t^3$

$$-19\sqrt{13}t^3-260t^2-91\sqrt{13} \cdot t-169 = 0;$$

that is,

$$(t^3-1)\{169t^6+91\sqrt{13}.t^5+29t^4+110\sqrt{13}.t^3 \\ +429t^2+91\sqrt{13}.t+169\}=0.$$

In the latter factor put

$$y=\sqrt{13}\left(t+\frac{1}{t}\right)=r+\frac{13}{r};$$

then $$y^3+7y^2-6y-72=0,$$

or $$(y-3)(y+4)(y+6)=0.$$

Thus the equations for r are

$$r^2 \quad -13=0,$$
$$r^2-3r+13=0,$$
$$r^2+4r+13=0,$$
$$r^2+6r+13=0.$$

(iii.) The remaining equation in t is reciprocal, and as follows:—

$$28561t^{16}+30758\sqrt{13}.t^{15}+195533t^{14}+53742\sqrt{13}.t^{13} \\ +113061t^{12}+13364\sqrt{13}.t^{11}+71890t^{10} \\ +44036\sqrt{13}.t^9+208510t^8+\ldots=0.$$

If we write $r+\frac{13}{r}=y$, as before, we find

$$y^8+14y^7-15y^6-956y^5-2893y^4+13482y^3 \\ +70959y^2+57348y-43092=0;$$

that is,

$$(y+2)(y+7)(y^2-27)(y^2-y-38)(y^2+6y-3)=0.$$

Hence the remaining equations for r are

$$r^2+2r \quad +13=0,$$
$$r^2+7r \quad +13=0,$$
$$r^2\pm 3\sqrt{3}.r+13=0,$$
$$r^2-\left(\frac{1\pm 3\sqrt{17}}{2}\right)r+13=0,$$
$$r^2+(3\pm 2\sqrt{3})\,r+13=0.$$

All the values of r, and the corresponding values of J, are contained in the following table:—

$$r = \pm\sqrt{13},$$

$$J = \frac{5^3}{2^3.3^3}(\pm\sqrt{13}-2)(\pm\sqrt{13}+4)(\pm\sqrt{13}+5)^3(\pm\sqrt{13}+6)^3$$

$$= \frac{5^3}{2^3}(\pm\sqrt{13}+3)^3(\pm\sqrt{13}+6)^3,$$

$$r = \frac{-5\pm 3i\sqrt{3}}{2}, \quad J = 0,$$

$$r = \frac{3\pm i\sqrt{43}}{2}, \quad J = -2^{12}.5^3,$$

$$r = -2\pm 3i, \quad J = \frac{11^3}{2^3},$$

$$r = -3\pm 2i, \quad J = 1,$$

$$r = -1\pm 2i\sqrt{3}, \quad J = -\frac{2^9.5^3}{3^3},$$

$$r = \frac{-7\pm i\sqrt{3}}{2}, \quad J = \frac{5^3}{2^3},$$

$$r = \frac{3\sqrt{3}\pm 5i}{2}, \quad J = \frac{5^3}{2^4}(4-3\sqrt{3})^3(3-2\sqrt{3})^3,$$

$$r = \frac{-3\sqrt{3}\pm 5i}{2}, \quad J = \frac{5^3}{2^4}(4+3\sqrt{3})^3(3+2\sqrt{3})^3,$$

$$r = \left(\frac{1\pm i\sqrt{51}}{2}\right)\left(\frac{1\mp i\sqrt{3}}{2}\right), \quad J = -2^3(7-\sqrt{17})(4+\sqrt{17})^4,$$

$$r = \left(\frac{1\pm i\sqrt{51}}{2}\right)\left(\frac{1\pm i\sqrt{3}}{2}\right), \quad J = -2^3(7+\sqrt{17})(4-\sqrt{17})^4.$$

$$r = (3\pm 2i)\left(\frac{-1\pm i\sqrt{3}}{2}\right),$$

$$J = -\frac{\sqrt{3}}{2^3.3^3}(\sqrt{3}-1)(\sqrt{3}-1)^3(3\sqrt{3}-2)^3(2\sqrt{3}-1)^3$$

$$= -\frac{\sqrt{3}}{2^3.3^3}(\sqrt{3}-1)^4(3\sqrt{3}-2)^3(2\sqrt{3}-1)^3,$$

$$r = (3\pm 2i)\left(\frac{-1\mp i\sqrt{3}}{2}\right),$$

$$J = \frac{\sqrt{3}}{2^3.3^4}(\sqrt{3}+1)^4(3\sqrt{3}+2)^3(2\sqrt{3}+1)^3.$$

The corresponding classes are

$$(1, 0, 13), \quad (2, 1, 7), \quad (2, 7, 26) \sim (2, 1, 2),$$

$$(2, 3, 26) \sim (2, 1, 22), \quad (1, 3, 13) \sim (1, 0, 4),$$

$$(1, 0, 1), \quad (2, 5, 26) \sim (2, 1, 14), \quad (1, 0, 3),$$

$$(1, 1, 13) \sim (1, 0, 12), \quad (3, 0, 4), \quad (2, 1, 26),$$

$$(6, 3, 10), \quad (2, 1, 5), \quad (1, 2, 13) \sim (1, 0, 9).$$

I may add that I have verified almost all the foregoing results by comparison with those given in the papers already quoted, as well as in Weber's memoirs in the *Acta Mathematica*, Vols. VI. and XI.

Thursday, March 13th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

Perfect Numbers: Major Macmahon, R.A.

The Relation of Distortion in Prismatic Images to Dispersion: Dr. J. Larmor.

On the Satellite of a Line relatively to a Cubic: The President (Prof. Greenhill in the Chair).

An Approximate Relation connecting Successive Terms of the Expansion for $\tan x$: G. Heppel.

The following presents were received:—

"Proceedings of the Royal Society," Nos. 285, 286.

"Educational Times," for March.

"Proceedings of the Physical Society of London," Vol. X., Part III.

"Proceedings of the Royal Irish Academy," Third Series, Vol. I., No. 2; Dublin, 1889.

"Transactions of the Royal Irish Academy," Vol. XXIX., Part XII.; Dublin, 1889.

"Chess Telegraphic Codes," by Edwin Anthony, 4to pamphlet; London, 1890.

"Annual Report of the Canadian Institute," Session 1888-9; 8vo; Toronto, 1889.

"Bulletin des Sciences Mathématiques," Tome XIV., Fev. 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 2.

"Rendiconti del Circolo Matematico di Palermo," Fasc. 1 e 2; 1890.
"Atti della R. Accademia dei Lincei—Rendiconti," Vol. v., Fasc. 11–13.
"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 99, 100.
"Acta Mathematica," xiv., 1.
"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxiv., Livr. 1re.
"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. iii., Fasc. 1–12; Napoli.
"American Journal of Mathematics," Vol. xii., No. 3; Baltimore, March, 1890.
"Haverford College Studies," published by the Faculty of Haverford College, No. 1, Pa.

On the Satellite of a Line relatively to a Cubic.

By J. J. Walker, M.A., F.R.S.

[*Read March 13th*, 1890.]

Introduction.

1. In a "Memoir," of some length, "On the Diameters of a Plane Cubic" (*Phil. Trans.*, 1888 A., pp. 166–170), the equation of the Satellite of a line was incidentally arrived at in a form (48)(38)(37) [(25) of the present paper], involving the square of the primitive line as an implicit factor. It did not come within the scope of that memoir to discuss the reduction of this form with the generality which its importance authorised, nor were the subsidiary results, on which that general reduction is based, at that time published. I therefore limited myself to verifying the reduction given by reference to special forms of the cubic.

The results referred to having since appeared, as quoted hereinafter, I show in the present paper how the general reduction may be effected; the method of which appears to have some novelty and interest *per se*; and to save the necessity of reference to the "Memoir," as well as to obtain the equation in a slightly simplified form at once —viz., one in which only the first power of the line is implicitly involved—I revert to what is, in fact, the original investigation by which I arrived at it. Four forms of the Satellite Line are deduced (24), (25), (26), (27), in the last two of which the product of the "Quippian" *of u and the line L* enters.

2. Let

$$u \equiv a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3a_3x^2z+3b_1y^2x+3b_3y^2z+3c_1z^2x$$
$$+3c_2z^2y+6exyz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

be the equation of the plane cubic, and

$$3u_1=\partial u/\partial x, \quad 3u_2=\partial u/\partial y, \quad 3u_3=\partial u/\partial z \quad \ldots\ldots\ldots\ldots(2),$$

$$\left.\begin{aligned} 6a=\partial^2 u/\partial x^2, \quad 6b=\partial^2 u/\partial y^2, \quad 6c=\partial^2 u/\partial z^2 \\ 6f=\partial^2 u/\partial y\partial z, \ 6g=\partial^2 u/\partial z\partial x, \ 6h=\partial^2 u/\partial x\partial y \end{aligned}\right\} \ldots\ldots\ldots(3).$$

Further, let $$L \equiv lx+my+nz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

be any linear form; then

$$s \equiv (bc-f^2)\,l^2+\ldots+2\,(gh-af)\,mn+\ldots \quad \ldots\ldots\ldots\ldots\ldots(5)$$

will be the envelope of polar lines of points on $L=0$, or, briefly, the "Poloid" of L. If s be written in the normal form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

then, as I have pointed out in the "Memoir,"

$$a=0, \quad b=0, \quad c=0$$

are the reciprocals of u_1, u_2, u_3, and

$$2f=0, \quad 2g=0, \quad 2h=0$$

the contravariant conics of u_2u_3, u_3u_1, u_1u_2, respectively; these coefficients being connected by the relations

$$\left.\begin{aligned} \frac{\partial a}{\partial l}+\frac{\partial h}{\partial m}+\frac{\partial g}{\partial n} \equiv 0, \\ \frac{\partial h}{\partial l}+\frac{\partial b}{\partial m}+\frac{\partial f}{\partial n} \equiv 0, \\ \frac{\partial g}{\partial l}+\frac{\partial f}{\partial m}+\frac{\partial c}{\partial n} \equiv 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and by those among their second differential coefficients which these involve, as

$$\frac{\partial^2 a}{\partial l^2}+\frac{\partial^2 h}{\partial l\partial m}+\frac{\partial^2 g}{\partial n\partial l} \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

or, as they (7, 8) may be more briefly written,

$$\left.\begin{aligned} 2a_1+2h_2+2g_3 &\equiv 0,\\ 2h_1+2b_2+2f_3 &\equiv 0,\\ 2g_1+2f_2+2c_3 &\equiv 0,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$1\,.\,2\,.\,a_{11}+1\,.\,2\,.\,h_{12}+1\,.\,2\,.\,g_{31}=0 \quad \ldots\ldots\ldots\ldots\ldots(10).$$

The condition that s should be touched by the line

$$\xi x+\eta y+\zeta z=0,$$

is written

$$\sigma \equiv A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0 \quad \ldots\ldots\ldots(11),$$

viz., $$A=bc-f^2\ldots,\quad F=gh-af\ldots \quad \ldots\ldots\ldots\ldots\ldots(12),$$

and the condition that $lx+my+nz=0$

should touch $u=0,$

$$5v \equiv A'l^2+\ldots+2F'mn+\ldots=0 \quad \ldots\ldots\ldots\ldots\ldots(13),$$

where $$6A'=\frac{\partial^2 v}{\partial l^2}\ldots \quad 6F'=\frac{\partial^2 v}{\partial m\partial n}\ldots \quad \ldots\ldots\ldots\ldots\ldots(14).$$

3. It is known that $v+4\sigma$ (in which ξ, η, ζ are replaced by l, m, n),

or, $$(A'+20A)\,l^2+\ldots+2\,(F+20F)\,mn+\ldots\equiv 0 \quad \ldots\ldots\ldots(15);$$

but $A'\ldots F'\ldots$ are expressed in terms of $A\ldots F\ldots l, m, n$ through the equations

$$A'=-4\left(A+l\frac{\partial A}{\partial l}+m\frac{\partial H}{\partial l}+n\frac{\partial G}{\partial l}\right) \quad \ldots\ldots\ldots\ldots\ldots(16),$$

$$\left.\begin{aligned} F'&=-4\left(F+l\frac{\partial H}{\partial n}+m\frac{\partial B}{\partial n}+n\frac{\partial F}{\partial n}\right)\\ &=-4\left(F+l\frac{\partial G}{\partial m}+m\frac{\partial F}{\partial m}+n\frac{\partial C}{\partial m}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(17),$$

with analogous formulæ for B', C', G', H', as I have shown [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 502 (95), (97)]; or in terms of l, m, n and the

coefficients of λ,

$$\left.\begin{aligned} A' &= -4\{3(bc-f^2)+(bc_{11}-cb_{11}-2ff_{11})l^2+\dots \\ &\qquad +2(gb_{11}-hg_{11}-af_{11}-fa_{11})mn+\dots\} \\ F' &= -4\{3(gh-af)+(bc_{23}-cb_{23}-2ff_{23})l^2+\dots \\ &\qquad +2(gh_{23}+hg_{23}-af_{23}-fa_{23})mn+\dots\} \end{aligned}\right\}\dots(18),$$

where, as in (10),

$$c_{11}=\partial^2 c/2\partial l^2\dots c_{23}=\partial^2 c/2\partial m\partial n\dots.$$

Equation of the Satellite Line.

4. If, now, $x'y'z'$ is a point in which

$$L\equiv lx+my+nz=0$$

meets the cubic $u=0$, the tangent at this point is (3)

$$ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=0;$$

and, since this also touches the Poloid of L (6, 11) if (xyz) is any other point on the tangent to u at $x'y'z'$,

$$A(z'y-y'z)^2+\dots+2F(x'z-z'x)(y'x-x'y)+\dots=0.$$

Eliminating $x'y'z'$ among the above equations, and

$$lx'+my'+nz'=0,$$

$$\{(bc'+cb'-2ff')l^2+\dots\}^2-4\{(bc-f^2)l^2+\dots\}\{(b'c'-f'^2)l^2+\dots\}=0 \qquad \dots\dots\dots\dots(19),$$

if $$b'-(b'^2+A'^2-2b'x\dots f'=-Ayz-Fx^2+Gxy+Hzx\dots\dots(20).$$

The result of these substitutions is

$$4\{(bc-f^2)l^2+\dots+2(gh-af)mn+\dots\}$$

is $$4\Delta^2\{(BC-F^2)x^2+\dots+2(GH-AF)yz+\dots\},$$

or $$4\Delta^2(abc+2fgh-af^2-bg^2-ch^2)(ax^2+\dots+2fyz+\dots)\dots\dots(21).$$

It was shown by Cayley (*Phil. Trans.*, 1857, p. 432), by means of the canonical form of u, and is proved generally in the "Memoir" (*Phil. Trans.*, 1888, A, p. 160), that

$$4(abc+2fgh-af^2-bg^2-ch^2)=I^2,$$

I being the "Hippian" of u.

Thus (5), (6), (21), the product

$$4\{(bc-f^2)\,l^2+\ldots\}\{(b'c'-f'^2)\,l^2+\ldots\}=P^2L^2s\ldots\ldots\ldots(22).$$

But, by the same substitutions (20),

$$(b'c+bc'-2ff')\,l^2+\ldots+2\,(fg'+f'g-ch'-c'h)\,lm$$

becomes (2), (3), identically

$$(Al^2+\ldots+2Fmn+\ldots)\,u$$
$$+L^2\,(Aa+Bb+Cc+2Ff+2Gg+Hh)$$
$$-2L\{(Al+Hm+Gn)\,u_1+(Hl+Bm+Fn)\,u_2+(Gl+Fm+Cn)\,u_3\}^*;$$

so that (22), the eliminant (19), is the product of the factors (w, v),

$$(Al^2+\ldots)\,u+L^2\,(Aa+\ldots)-L\left[2\{(Al+\ldots)\,u_1+\ldots\}\pm Ps\right]\ldots(23).$$

5. It will be shown below (p. 254) that the terms

$$2\{(Al+\ldots)\,u_1+(Hl+\ldots)\,u_2+(Gl+\ldots)\,u_3\}+Ps$$
$$\equiv\left\{2\,(Aa+\ldots+2Ff+\ldots)+\tfrac{1}{2}\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}\right\}L\ldots\ldots\ldots\ldots(\text{i.});$$

so that the first of the two factors (23) becomes, identically,

$$w=(Al^2+\ldots)\,u-\frac{L^2}{2}\left\{2\,(Aa+\ldots+2Ff+\ldots)+\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\ldots\right\};$$

viz., this is the equation of the three tangents at the points in which L meets u, the other factor (v) being (as will also be shown, p. 261) a proper (nodal) cubic concomitant of u and L.

Thus it appears that one form of the satellite of the line L is

$$2\,(Aa+Bb+Cc+2Ff+2Gg+2Hh)+\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}=0\ \ldots\ldots\ldots(24).$$

Other Forms of the Equation of the Satellite.

6. It will further be shown below (pp. 256–7) that (14)

$$-\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}\equiv L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2 P}{\partial m\,\partial n}+\ldots\right)$$
$$+\Sigma\,(A'+24A)\,a+2\Sigma\,(F'+24F)\,f\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

---

* Observe that $\{\ \}$ is the u-polar conic of the s-pole of L.

wherein the term, multiplied by L, the result of substituting symbols of differentiation with respect to l, m, n for x, y, z of s (6), and operating on P, is, plainly, a contravariant of u—or an invariant of L and u—of order 5 in the coefficients of u and order 3 in those of L: viz., it is, in fact, the "Quippian" Q, as may be verified by means of the canonical or other form of u [*Proc. Lond. Math. Soc.*, Vol. XIX., p. 491 (46)].

But it will also be shown (p. 259) that

$$-2\Sigma \frac{\partial P}{\partial l}\frac{\partial s}{\partial x} \equiv \Sigma (A'+20A)\,a+2\Sigma (F'+20F)f \ldots\ldots\ldots\text{(iii.)};$$

and from (ii.), (iii.) it follows that

$$-2L\left(a\frac{\partial^2 P}{\partial l^2}+\ldots\right) \equiv \Sigma (A'+28A)\,a+2\Sigma (F'+28F)f \ldots\ldots\text{(iv.)}.$$

Adding (iii.) to (24), multiplied by 2, a second form of the satellite line of L is obtained, viz.,

$$(A'+16A)\,a+\ldots+2(F'+16F)f+\ldots = 0 \ldots\ldots\ldots\ldots\text{(25)};$$

the form given in the "Memoir" [*Phil. Trans.*, A. 1888 (41), p. 167, and (49) p. 170.] Finally, eliminating successively

$$\Sigma A'a+2\Sigma F'f$$

and

$$\Sigma Aa+2\Sigma Ff$$

from the above equation to the satellite line by means of (iv.); viz.,

$$\Sigma A'a+2\Sigma F'f \equiv -28(\Sigma Aa+2\Sigma Ff)-2LQ,$$

that equation takes the forms

$$6(\Sigma Aa+2\Sigma Ff)+QL=0 \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

and

$$3(\Sigma A'a+2\Sigma F'f)-8QL=0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(27)}.$$

Pencil of Six Lines.

7. The last two equations show that the satellite line—say L'—of L, the line L and the two lines K, K', where

$$K \equiv Aa+\ldots+2Ff+\ldots$$

$$K' \equiv A'a+\ldots+2F'f+\ldots$$

form a pencil of four lines which all meet in the point whose co-

ordinates are determined by (3),

$$\begin{aligned}&\big[(Aa_1+Bb_1+Cc_1+2Fe+2Ga_3+2Ha_2)\,x\\&+(Aa_2+Bb_2+Cc_2+2Fb_3+2Ge+2Hb_1)\,y\\&+(Aa_3+Bb_3+Cc_3+2Fc_2+2Gc_1+2He)\,z=0,\ i.e.\big].\end{aligned}$$

$$\Theta_1 x+\Theta_2 y+\Theta_3 z=0,\quad lx+my+nz=0,$$

Θ_1, Θ_2, Θ_3 being the invariants Θ of s and u_1, s and u_2, s and u_3, respectively.

The special values for those coordinates, in the case of the canonical u, have been given by Cayley in the "Memoir on Cubic Curves" (*Phil. Trans.*, 1857, § 33).

It will be remarked that, if L touches the Quippian of u, the three lines L', K, K' all coincide.

Equation (iii.), above, shows that the s-polar of the point

$$x:y:z=\frac{\partial P}{\partial l}:\frac{\partial P}{\partial m}:\frac{\partial P}{\partial n}$$

is another line passing through the intersection of K, K', L, L'. A sixth line passing through the same common intersection is the u-polar line of the s-pole of L.

For the s-pole of L is the point whose coordinates are proportional to (11)

$$\frac{\partial\sigma^*}{\partial l},\ \frac{\partial\sigma}{\partial m},\ \frac{\partial\sigma}{\partial n},$$

and its u-polar line is, therefore,

$$a\left(\frac{\partial\sigma}{\partial l}\right)^2+\ldots+2f\left(\frac{\partial\sigma}{\partial m}\,\frac{\partial\sigma}{\partial n}\right)+\ldots=0.$$

But, generally, since the discriminant of $s=P^2/4$,

$$\left(\frac{\partial\sigma}{\partial l}\right)^2=4A\sigma-P^2\,(bn^2+cm^2-2fmn)\ldots,$$

$$\frac{\partial\sigma}{\partial m}\,\frac{\partial\sigma}{\partial n}=4F\sigma-P^2\,(amn+fl^2-glm-hnl)\ldots.$$

---

* *i.e.*, $\frac{\partial\sigma}{\partial\xi}$ with ξ, η, ζ replaced by l, m, n.

By which substitutions the equation above becomes

$$4(Aa+\ldots+2Ff+\ldots)\sigma - P^2\{(bn^2+cm^2-2fmn)a+\ldots \\ +2(-amn-fl^2+glm+hnl)f+\ldots\}.$$

Now (15), $4\sigma = -v,$

and it will be shown below that

$$(bn^2+cm^2-2fmn)a+\ldots \equiv -PL \ldots\ldots\ldots\ldots\ldots\ldots (v.);$$

so that, finally, the u-polar line of the s-pole of L is

$$v(Aa+\ldots+2Ff+\ldots)+P^2L=0.$$

Proof of Equation (v.), *above.*

8. Since [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 494 (62), (63), (64)]

$$-lP \equiv (bc_1+cb_1-2fe)l^2+\ldots$$
$$-mP \equiv (bc_2+cb_2-2fb_2)l^2+\ldots$$
$$-nP \equiv (bc_3+cb_3-2fc_3)l^2+\ldots,$$

with the notation of (1) ... (6) above,

$$-(lx+my+nz)P \equiv (bc+cb-2ff)l^2+\ldots \\ \equiv (bn^2+cm^2-2fmn)a+\ldots \\ +2(-amn-fl^2+glm+hnl)f+\ldots \Bigg\} \ldots\ldots (28).$$

Proof of Equation (i.), p. 251.

9. Dividing both sides of this identity by

$$L \equiv lx+my+nz,$$

and operating with

$$a\frac{\partial^2}{\partial l^2}+\ldots+2f\frac{\partial^2}{\partial m\,\partial n}+\ldots,$$

the dexter being treated as the product of L^{-1}, and the dexter of (28), in which it is to be observed a, b, c, f, g, h are quadric functions

of l, m, n;

$$-\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\mathrm{b}\frac{\partial^2 P}{\partial m^2}+\mathrm{c}\frac{\partial^2 P}{\partial n^2}+2\mathrm{f}\frac{\partial^2 P}{\partial m\partial n}+2\mathrm{g}\frac{\partial^2 P}{\partial n\partial l}+2\mathrm{h}\frac{\partial^2 P}{\partial l\partial m}\right)$$

$$\equiv 2L^{-3}s(-LP)+L^{-1}\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2}{\partial m\partial n}+\ldots\right)(-LP)$$

$$-4L^{-2}\left\{\begin{array}{r}(\mathrm{a}x+\mathrm{h}y+\mathrm{g}z)\{(\mathrm{bc}+\mathrm{cb}-2\mathrm{f}f)\,l+(\mathrm{f}g+\ldots)\,m+(\mathrm{h}f+\ldots)\,n\\ +(\mathrm{b}_1c+\mathrm{c}_1b-2\mathrm{f}_1f)\,l^2+\ldots\quad+\ldots\}\\ +(\mathrm{h}x+\mathrm{b}y+\mathrm{f}z)\{(\mathrm{f}g+\ldots-\mathrm{hc})\,l+(\mathrm{c}a+\ldots)\,m+(\mathrm{g}h+\ldots)\,n\\ +(\mathrm{b}_2c+\mathrm{c}_2b-2\mathrm{f}_2f)\,l^2+\ldots\quad+\ldots\}\\ +(\mathrm{g}x+\mathrm{f}y+\mathrm{c}z)\{(\mathrm{h}f+\ldots-\mathrm{g}b)\,l+(\mathrm{g}h+\ldots)\,m+(\mathrm{a}b+\ldots)\,n\\ +(\mathrm{b}_3c+\mathrm{c}_3b-2\mathrm{f}_3f)\,l^2+\ldots\quad+\ldots\}\end{array}\right.$$

$$\ldots\ldots\ldots\ldots(29),$$

the same notation being employed as above (9), viz.:

$$\mathrm{b}_1=\partial\mathrm{b}/2\partial l,\quad \mathrm{b}_2=\partial\mathrm{b}/2\partial m\ \ldots\ \mathrm{f}_3=\partial\mathrm{f}/2\partial n\ \ldots\ .$$

But (6)

$$\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots\right)(LP)=L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\ldots\right)+\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\frac{\partial}{\partial m}\frac{\partial s}{\partial y}+\frac{\partial P}{\partial n}\frac{\partial s}{\partial z},$$

and the sum of the first, third, and fifth lines of the terms multiplied by $-4L^{-2}$, when multiplied out, will be found to be identically equal to [(2), (12)]

$$Aa+Bb+Cc+2Ff+2Gg+2Hh$$

$$-\{(Al+Hm+Gn)\,u_1+(Hl+Bm+Fn)\,u_2+(Gl+Fm+Cn)\,u_3\};$$

also [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 493, (59), (65)],

$$\left.\begin{array}{l}(\mathrm{b}_1c+\mathrm{c}_1b-2\mathrm{f}_1f)\,l^2+\ldots+2\,(\mathrm{g}_1h+\mathrm{h}_1g-\mathrm{a}_1f-\mathrm{f}_1a)\,mn+\ldots=-xP,\\ (\mathrm{b}_2c+\mathrm{c}_2b-2\mathrm{f}_2f)\,l^2+\ldots+2\,(\mathrm{g}_2h+\mathrm{h}_2g-\mathrm{a}_2f-\mathrm{f}_2a)\,mn+\ldots=-yP,\\ (\mathrm{b}_3c+\mathrm{c}_3b-2\mathrm{f}_3f)\,l^2+\ldots+2\,(\mathrm{g}_3h+\mathrm{h}_3g-\mathrm{a}_3f-\mathrm{f}_3a)\,mn+\ldots=-zP,\end{array}\right\}$$

so that the sum of the second, fourth, and sixth lines of the terms in (29), multiplied by $-4L^{-2}$, is simply equal to

$$-Ps.$$

Making these substitutions, (29) becomes

$$-L^{-1}\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}-4L^{-2}\{L(Aa+\ldots+2Ff+\ldots)$$

$$-(Al+\ldots)u_1-(Hl+\ldots)u_2-(Gl+\ldots)u_3\}+2L^{-2}Ps\equiv 0,$$

which, multiplied by $L^2/2$, is the identity to be proved.

Proof of Identity (ii.), p. 251.

10. Returning to the identity (28),

$$-LP\equiv(\mathrm{bc}+\mathrm{cb}-2\mathrm{f}f)\,l^2+\ldots+2(\mathrm{g}h+\mathrm{h}g-\mathrm{a}f-\mathrm{f}a)\,mn+\ldots$$

$$\equiv(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\,a+(\mathrm{c}l^2+\mathrm{a}n^2-2\mathrm{g}nl)\,b+(\mathrm{a}m^2+\mathrm{b}l^2-2\mathrm{h}lm)c$$

$$+2(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)f+2(-\mathrm{b}nl+\ldots)\,g+2(-\mathrm{c}lm+\ldots)h$$

.....(30),

and, operating on this with

$$\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2}{\partial m\partial n}+\ldots,$$

the result on the sinister is

$$-L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\mathrm{b}\frac{\partial^2 P}{\partial m^2}+\mathrm{c}\frac{\partial^2 P}{\partial n^2}+2\mathrm{f}\frac{\partial^2 P}{\partial m\partial n}+2g\frac{\partial^2 P}{\partial n\partial l}+2\mathrm{h}\frac{\partial^2 P}{\partial l\partial m}\right)$$

$$-\left(\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\ldots\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

if (6) $$s\equiv\mathrm{a}x^2+\ldots+2\mathrm{f}yz+\ldots.$$

For the result on the dexter side there is the choice of three forms (*Lond. Math. Soc. Proc.*, Vol. xx., p. 111), of which that ("γ," *ibid.* and p. 116) most suited to the present case is as follows:—

If v, w are two quadric forms in l, m, n [as are b, n^2 ... in (30)],

then $$\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots 2\mathrm{f}\frac{\partial^2}{\partial m\,\partial n}+\ldots\right)vw$$

$$=12\{\mathrm{a}v_1w_1+\ldots\qquad+\mathrm{f}(v_2w_3+v_3w_2)+\ldots\}$$

$$+2\{(\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)\,l^2+\ldots+2(\mathrm{g}H_1+\mathrm{h}G_1-\mathrm{a}F_1-\mathrm{f}A_1)\,mn+\ldots\}$$

.....(vi.).

where

$$v_1 = \partial v/2\partial l, \quad v_2 = \partial v/2\partial m \ldots w_3 = \partial w/2\partial n,$$

$$A_1 = \tfrac{1}{4}\left(\frac{\partial^2 v}{\partial m^2}\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 v}{\partial n^2}\frac{\partial^2 w}{\partial m^2} - 2\frac{\partial^2 v}{\partial m \partial n}\frac{\partial^2 w}{\partial m \partial n}\right),$$

$$F_1 = \tfrac{1}{4}\left(\frac{\partial^2 v}{\partial n \partial l}\frac{\partial^2 w}{\partial l \partial m} + \frac{\partial^2 w}{\partial n \partial l}\frac{\partial^2 v}{\partial l \partial m} - \frac{\partial^2 v}{\partial l^2}\frac{\partial^2 w}{\partial m \partial n} - \frac{\partial^2 w}{\partial l^2}\frac{\partial^2 v}{\partial m \partial n}\right),$$

with analogous values for v_3, w_1, w_2, B_1, C_1, G_1, H_1.

The result of the operation on the dexter of (30) is, then,

$$\begin{aligned}
&12\Sigma a\{2\,(bc-f^2)\\
&-l\,(bc_1+cb_1-2ff_1)-m\,(fg_1+gf_1-ch_1-hc_1)-n(hf_1+fh_1-bg_1-gb_1)\}\\
&+12\Sigma f\{4\,(gh-2f)\\
&-l\,(fg_3+gf_3-ch_3-hc_3)-m\,(ca_3+ac_3-2gg_3)-n\,(gh_3+hg_3-af_3-fa_3)\\
&-l(hf_2+fh_2-bg_2-gb_2)-m(gh_2+hg_2-af_2-fa_2)-n(ab_2+ba_2-2hh_2)\}\\
&+\ 2\Sigma a\{(bc_{11}+cb_{11}-2ff_{11})\,l^2+\ldots+2\,(gh_{11}+\ldots)\,mn+\ldots\}\\
&+\ 4\Sigma f\{(bc_{23}+cb_{23}-2ff_{23})\,l^2+\ldots+2\,(gh_{23}+\ldots)\,mn+\ldots\} \quad \ldots\ldots\ldots(32).
\end{aligned}$$

[In forming the above result, consider the multiplier of a,

$$bn^2+cm^2-2fmn,$$

in the operand (30). For it, plainly,

$$\begin{aligned}
av_1w_1+\ldots+f\,(v_2w_3+v_3w_2)+\ldots &= b\,(mc_2-nf_2)+c\,(nb_3-mf_3)\\
&+f\,(nb_2+mc_3-mf_3-nf_3)+g\,(nb_1-mf_1)+h\,(mc_1-nf_1)\,;\\
\text{i.e., (9),}\quad &= b\,(c-lc_1+ng_1)+c\,(b-lb_1+mh_1)\\
&+f(-2f+2lf_1-nh_1-mg_1)+g\,(nb_1-mf_1)+h\,(mc_1-nf_1)\\
&= 2\,(bc-f^2)\\
&-l\,(bc_1+cb_1-2ff_1)-m\,(gf_1+fg_1-ch_1-hc_1)-n\,(hf_1+fh_1-bg_1-gb_1),
\end{aligned}$$

which is the sum of terms multiplied by a in the first two lines of (32); and

$$A_1 = b_{22}+c_{33}+2f_{23} = (9)-h_{12}-g_{31} = a_{11},\quad B_1 = b_{11},\quad C_1 = c_{11},\quad F_1 = f_{11},$$

$$G_1 = -f_{12}-c_{31} = g_{11},\quad H_1 = -b_{12}-f_{23} = h_{11}\,;$$

whence, (vi.),

$$(bC_1+cB_1-2fF_1)l^2+\ldots=(bc_{11}+cb_{11}-2ff_{11})l^2+(ca_{11}+ac_{11}-2gg_{11})m^2+\ldots$$
$$+2\,(gh_{11}+hg_{11}-af_{11}-fa_{11})\,mn+\ldots,$$

agreeing with the sixth line of (32).

Again, for the part which multiplies f in (30), viz.,

$$-amn-fl^2+glm+hln,$$

$$av_1w_1+\ldots=a\,(-2lf_1+mg_1+nh_1)+b\,(-na_3+lg_3)+c\,(-ma_3+lh_3)$$
$$+f\,(-ma_2-na_3+lg_3+lh_2)$$
$$+g\,(-ma_1-2lf_3+mg_3+nh_3+lh_1)$$
$$+h\,(-na_1-2lf_3+lg_1+mg_3+nh_2);$$

i.e., (9), $=a\,(-2f-mc_3-nb_3+m_1f_2+mf_3)+b\,(\ldots)+c\,(\ldots)$
$$+f\,(-2a-lh_3-lg_3+ma_3+na_3)$$
$$+g\,(2h+lb_3-lf_3+2mg_3-mh_3-nh_3)$$
$$+h\,(2g+lc_3-lf_2-mg_2+2nh_2-ng_3)$$
$$=4\,(gh-af)-l\,(gf_3+fg_3+hf_2+fh_2-ch_3-hc_3-bg_2-gb_2)$$
$$-m\,(ca_3+ac_3+gh_2+hg_2-2gg_3-af_3-fa_3)$$
$$-n\,(ab_3+ba_3+gh_3+hg_3-2hh_3-af_3-fa_3);$$

and for the same,

$$A_1=2a_{23},\quad B_1=-2\,(h_{31}+f_{33})=2b_{23},\quad C_1=-2\,(g_{12}+f_{22})=2c_{23},$$
$$F_1=2f_{23}+a_{11}+h_{12}+g_{31}=2f_{23},\quad G_1=g_{23}-a_{12}-h_{22}=2g_{23},$$
$$H_1=h_{23}-a_{31}-g_{33}=2h_{23}.]$$

But [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 502 (95), (97)] the coefficient of a, in (32), is equal to [(12), (14), p. 249]

$$24A+3\,(A'+4A)/2-(A'+12A)/2$$
$$=A'+24A;$$

and, similarly, the coefficient of f

$$=2\,(F'+24F);$$

viz., (31), (32),

$$-L\left(a\frac{\partial^2P}{\partial l^2}+\ldots+2f\frac{\partial^2P}{\partial m\partial n}+\ldots\right)-\Sigma\left(\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}\right)$$
$$=\Sigma\left\{(A'+24A)\,a+2\,(F'+24F)f\right\}\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Corollary.

11. If the "crude" [*Proc.*, Vol. xx., p. 115 (15)] form of the result of the operation on the dexter of (30) is used, it is (as above, pp. 256–8),

$$4\{(bc-f^2)\,a+\ldots+2(gh-af)f+\ldots\}+8(av_1w_1+\ldots)$$
$$+2\Sigma a\{(b_{11}c+c_{11}b-2f_{11}f)\,l^2+\ldots\}$$
$$+4\Sigma f\{(b_{23}c+c_{23}b-2f_{23}f)\,l^2+\ldots\}$$

i.e., $4(Aa+\ldots+2Ff+\ldots)+(20A+A')\,a+\ldots+2(20F+F')f+\ldots$
$$+2\Sigma a\{(b_{11}c+\ldots)\,l^2+\ldots\}+4\Sigma f\{(b_{23}c+\ldots)\,l^2+\ldots\}$$

or [(14), p. 249] $\Sigma(A'+24A)\,a+2\Sigma(F'+24F)f+2\Sigma a\{\ldots\}+4\Sigma f\{\ldots\}$,

and comparing this result with (33), it appears that

$$\Sigma a\{(b_{11}c+c_{11}b-2f_{11}f)\,l^2+\ldots\}+2\Sigma f\{(b_{23}c+c_{23}b-2f_{23}f)\,l^2+\ldots\}\equiv 0$$
.....(vii.),

a result which will presently be of use, and may be verified independently, with a moderate amount of work, by expressing the formula as (*Lond. Math. Soc. Proc.*, Vol. ix., pp. 232, 3)

$$\Sigma x\left[\Sigma\{u_1D^2u_1-(Du_1)^2\}\{s_{11}D^2u_1+u_1D^2s_{11}-2Du_1Ds_{11}\}\right.$$
$$\left.+\Sigma\{u_2D^2u_2+u_3D^2u_3-2Du_2Du_3\}\{s_{23}D^2u_1+u_1D^2s_{23}-2Du_1Ds_{23}\}\right]/\Delta^2,$$

where $s_{11}=\frac{1}{2}\dfrac{\partial^2 s}{\partial l^2}$ ....., $s_{23}=\frac{1}{2}\dfrac{\partial^2 s}{\partial m\partial n}$ ..... .

Proof of Identity (iii.), p. 252.

12. Differentiating the triad of equations, p. 255, end, with respec to l, then multiplying by a, h, g, respectively; with respect to m, the multiplying by h, b, f, respectively; with respect to n, then multiplying by g, f, c, respectively; adding the nine results, and having regard for the identity with zero (vii.) just proved:

$$-2\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}$$

$$=8a\{b_1c+c_1b-2f_1f)\,l+(f_1g+g_1f-c_1h-h_1c)\,m$$
$$+(h_1f+f_1h-b_1g-g_1b)\,n\}$$

$$+8b\{(f_2g+g_2f-c_2h-h_2c)\,l+(c_2a+a_2c-2g_2g)\,m+(g_2h+h_2g-a_2f-f_2a)\,n\}$$
$$+8c\{(h_2f+f_2h-b_2g-g_2b)\,l+(g_2h+h_2g-a_2f-f_2a)\,m+(a_2b+b_2a-2h_2h)\,n\}$$
$$+8f\{(f_2g+g_2f-c_2h-h_2c)\,l+(c_2a+a_2c-2g_2g)\,m+(g_2h+h_2g-a_2f-f_2a)\,n$$
$$+(h_2f+f_2h-b_2g-g_2b)\,l+(g_2h+h_2g-a_2f-f_2a)\,m+(a_2b+b_2a-2h_2h)\,n\}$$
$$+8g\{(h_1f+f_1h-b_1g-g_1b)\,l+(g_1h+h_1g-a_1f-f_1a)\,m+(a_1b+b_1a-2h_1h)\,n$$
$$+(b_2c+c_2b-2f_2f)\,l+(f_2g+g_2f-c_2h-h_2c)\,m+(h_2f+f_2h-b_2g-g_2b)\,n\}$$
$$+8h\{(b_2c+c_2b-2f_2f)\,l+(f_2g+g_2f-c_2h-h_2c)\,m+(h_2f+f_2h-b_2g-g_2b)\,n$$
$$+(f_1g+g_1f-c_1h-h_1c)\,l+(c_1a+a_1c-2g_1g)\,m+(g_1h+h_1g-a_1f-f_1a)\,n\}$$
$$=8\Sigma a\{b\,(c_2m-f_2n)+c\,(b_2n-f_2m)+f\,(b_2n+c_2m-f_2m-f_2n)$$
$$+g\,(b_1n-f_1m)+h\,(c_1m-f_1n)\}$$
$$+8\Sigma f\{a\,(-2f_1l+g_1m+h_1n)+b\,(g_2l-a_2n)+c\,(h_2l-a_2m)$$
$$+f\,(g_2l-a_2n+h_2l-a_2m)+g\,(h_1l-a_1m-2f_2l+g_2m+h_2n)$$
$$+h\,(g_1l-a_1n-2f_2l+g_2m+h_2n)\},$$

which, pp. 257, 258, has been shown to be equal to

$$8\Sigma a\{2\,(bc-f^2)$$
$$-l\,(bc_1+cb_1-2ff_1)-m\,(fg_1+gf_1-ch_1-hc_1)-n\,(hf_1+fh_1-bg_1-gb_1)\}$$
$$+8\Sigma f\{4\,(gh-af)$$
$$-l\,(fg_2+gf_2-ch_2-hc_2)-m\,(ca_2+ac_2-2gg_2)-n\,(gh_2+hg_2-af_2-fa_2)$$
$$-l(hf_2+fh_2-bg_2-gb_2)-m(gh_2+hg_2-af_2-fa_2)-n(ab_2+ba_2-2hh_2)\};$$

$$[(12),\text{ p. }249]=\Sigma a\left(16A-4l\frac{\partial A}{\partial l}-4m\frac{\partial H}{\partial l}-4n\frac{\partial G}{\partial l}\right)$$
$$+2\Sigma f\left(16F-2l\frac{\partial H}{\partial n}-2m\frac{\partial B}{\partial n}-2n\frac{\partial F}{\partial n}\right.$$
$$\left.-2l\frac{\partial G}{\partial m}-2m\frac{\partial F}{\partial m}-2n\frac{\partial C}{\partial m}\right)$$

$$[(16),\text{ p. }249]=\Sigma a\,(A'+20A)+2\Sigma f\,(F'+20F).$$

On the factor (*v*) *of the Eliminant* (23), p. 251.

13. It may now be shown that

$$v = (Al^2+\ldots)\,u + L^2\,(Aa+\ldots)$$
$$-L\left[2\left\{(Al+\ldots)\,u_1+(Hl+\ldots)\,u_2+(Gl+\ldots)\,u_3\right\}-Ps\right]$$

is a cubic—the "Cotesian" of the "Memoir"—having a node at the s-pole of L. For [(1), (3), p. 248]

$$\frac{\partial v}{\partial x} = 3\,(Al^2+\ldots)\,u_1 + L^2\,(Aa_1+Bb_1+Cc_1+2Fe+2Ga_2+2Ha_3)$$
$$+L\Big[2l\,(Aa+\ldots+2Ff+\ldots)$$
$$-4\left\{(Al+\ldots)\,a+4\,(Hl+\ldots)\,h+4\,(Gl+\ldots)\,g\right\}+P\frac{\partial s}{\partial x}\Big]$$
$$+l\left[-2\left\{(Al+\ldots)\,u_1+(Hl+\ldots)\,u_2+(Gl+\ldots)\,u_3\right\}+Ps\right].$$

Substituting the coordinates of the s-pole of L, viz.,

$$\left.\begin{aligned} &Al+Hm+Gn \text{ for } x,\\ &Hl+Bm+Fn \text{ for } y,\\ &Gl+Fm+Cn \text{ for } z, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

L becomes $Al^2+\ldots+2Fmn+\ldots$,

and

$(Al+\ldots)\,a+(Hl+\ldots)\,h+(Gl+\ldots)\,g$ becomes identical with u_1,

while u_1 becomes, identically,

$$(Al^2+\ldots)\,(Aa_1+\ldots+2Ha_2)-D\left\{(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{f}_1e)\,l^2+\ldots\right\},$$

where D is the discriminant of s or $P^2/4$, and [*Proc.*, Vol. XIX., p. 494, (62)]

$$(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{f}e)\,l^2+\ldots+2\,(\mathrm{g}a_2+\mathrm{h}a_3-\mathrm{a}e-\mathrm{f}a_1)\,mn+\ldots = -lP.$$

Thus, in $\partial v/\partial x$,

$$3\,(Al^2+\ldots)\,u_1-4L\left\{(Al+\ldots)\,a+(Hl+\ldots)\,h+(Gl+\ldots)\,g\right\}$$
$$+L^2\,(Aa_1+\ldots+2Fe+\ldots)$$

becomes $$-l\,(Al^2+\ldots)\,DP \quad \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Again [(i.), p. 251], for all values of x, y, z,

$$2L\,(Aa+\ldots+2Ff+\ldots)-2\left\{(Al+\ldots)\,u_1+(Hl+\ldots)\,u_2+(Gl+\ldots)\,u_3\right\}$$
$$= Ps-\frac{L}{2}\,\Sigma\frac{\partial P}{\partial l}\frac{\partial P}{\partial s},$$

so that the residue of terms in $\partial v/\partial x$ is, for all values of x, y, z,

$$LP\frac{\partial s}{\partial x}+2lP_s-l\frac{L}{2}\,\Sigma\frac{\partial P}{\partial l}\frac{\partial P}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(36).}$$

For the present values (34) of x, y, z,

$$\frac{\partial s}{\partial x}=2Dl,$$

$$\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}=6DlP,$$

$$s=(Al^2+\ldots)\,D.$$

Thus the residue (36) becomes

$$2\,(Al^2+\ldots)\,lDP+2\,(Al^2+\ldots)\,lDP-3\,(Al^2+\ldots)\,lDP,$$

or

$$l\,(Al^2+\ldots)\,DP;$$

whence, and from (35), it appears that, for the values (34),

$$\frac{\partial v}{\partial x}=0.$$

Similarly, it may be proved that $\partial v/\partial y$, $\partial v/\partial z$ vanish for the same values (33) of x, y, z.

Thursday, April 3rd, 1890.

J. J. Walker, Esq., F.R.S., President, in the Chair.

Mr. E. J. Brooksmith, M.A., LL.M., late Scholar and Law Student of St. John's College, Cambridge, Instructor in Mathematics at the Royal Academy, Woolwich, was elected a member.

The following communications were made:—

On the Properties of some Circles connected with a Triangle formed by Circular Arcs: R. Lachlan, M.A.

Some Properties of Numbers: Mr. R. W. D. Christie.

The Modular Equations for $n = 17, 29$: R. Russell, M.A. (Communicated by Professor Greenhill.)

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVII., No. 287.
"Proceedings of the Cambridge Philosophical Society," Vol. VII., Part I.
"Educational Times," for April.
"Bulletin des Sciences Mathématiques," Tome XIV.; March, 1890.
"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 3.
"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo III., No. 3; Mexico, 1889.
"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 101.
"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 4; Coimbra, 1889.
"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 1, 2, 3.
"Annali di Matematica pura ed applicata," Tome XVII., Fasc. 4.
"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XXXIX.-XLIII.
"Bulletin International de l'Academie des Sciences de Cracovie—Comptes Rendus des Séances de l'Année 1890," Février, 1890.

On the Properties of some Circles connected with a Triangle formed by Circular Arcs. By R. LACHLAN, M.A.

[*Read April 3rd*, 1890.]

Introduction, §§ 1, 2.

1. Let ABC be a triangle formed by three given circular arcs, and let the complete circles be drawn intersecting again in A', B', C'; we obtain three triangles $A'BC$, $AB'C$, and ABC' which may be called the associated triangles of ABC, and four triangles $A'B'C'$, $AB'C'$, $A'BC'$, $A'B'C$ which are the inverse triangles with respect to the circle, which cuts the given circles orthogonally, of ABC and its associated triangles respectively.

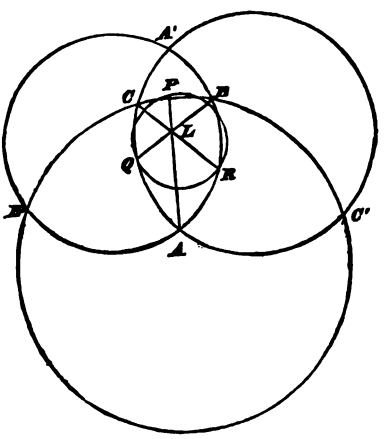
Each of these triangles has a circum-circle, an inscribed circle, and a Hart-circle, which last corresponds to the nine-point circle of an ordinary triangle. In the present paper I propose to exhibit the properties of these groups of circles in as complete a form as possible, but as some of these properties have already been investigated in a paper published in the *Quarterly Journal*, Vol. XXI., pp. 1–59, and in my Memoir on "Circles and Spheres," in the *Phil. Trans.*, Vol. CLXXVII., Part II., pp. 481–625, I shall merely quote results to be found in those papers.*

* References to these papers will be indicated by the letters *Q.*, *T.*, respectively.

Descriptive properties, viz., those involving only the angles at which the circles intersect, must obviously be true, whether the fundamental triangle is formed by three plane circular arcs or three spherical circular arcs. Also, since formulæ involving the radii of coplanar circles can always be deduced from the general formula (*cf.* *T.*, § 6 and § 129),

$$\pi \begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = 0,$$

connecting the powers of two systems of circles, which is the same for spherical circles, it follows that any formula involving the radii of plane circles can be transformed into a formula involving radii of spherical circles by substituting the tangents of the spherical radii for the radii of the plane circles. It will be sufficient, then, to confine our attention to coplanar circles.



2. From the figure we see that, if α, β, γ are the angles of the fundamental triangle ABC, the angles of the associated triangle $A'BC$ are $\alpha, \pi-\beta, \pi-\gamma$; and that the angles of any pair of inverse triangles are the same. As pointed out in *Q.*, § 41, from any formula for the triangle ABC, corresponding formulæ may be found for the associated triangles by changing α, β, γ according to the scheme—

ABC	α	β	γ
$A'BC$	α	$\pi-\beta$	$\pi-\gamma$
$AB'C$	$\pi-\alpha$	β	$\pi-\gamma$
ABC'	$\pi-\alpha$	$\pi-\beta$	γ

But to obtain from a formula for any triangle ABC the corresponding formula for the inverse triangle $A'B'C'$, we must take into account the position of the orthogonal circle; in fact, if O be the centre of the orthogonal circle, we have (see *Q.*, § 5) the theorem—

If the centre of inversion O lie within or without each of two given circles, the angle at which they intersect is equal to the angle at which the inverse circles intersect; but if O lie within one circle and without the other, the angle at which they intersect is the supplement of the angle at which the inverse circles intersect.

In the sequel, unless otherwise stated, we shall suppose the radical centre of the given circles to lie within the triangle ABC, and the orthogonal circle of the given circles to be imaginary.

The Group of In-Circles, §§ 3, 4.

3. It has been proved in *T.*, § 45, that the in-circles of the eight triangles are touched by eight other circles; the theorem may be stated thus:—

The in-circles of any triangle and the in-circles of the three associated triangles are touched by another circle, which touches the first internally and the rest externally.

Thus each triangle has a circle corresponding to the nine-points circle of a plane rectilinear triangle,—it is called the Hart-circle.

It is easily proved (*T.*, § 46) that the Hart-circle of any triangle, whose angles are α, β, γ, cuts the sides at angles $\beta \sim \gamma$, $\gamma \sim \alpha$, $\alpha \sim \beta$.

4. If the inscribed circle of the triangle ABC cut the orthogonal circle at the angle ω, we have (*T.*, § 43),

$$K \cos^2 \omega = 2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma),$$

where

$$K = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1.$$

Also, if r_1, r_2, r_3 are the radii of the given circles, and r the radius of the orthogonal circle, we have (*T.*, § 44) for the radius of the inscribed circle,

$$\begin{vmatrix} \frac{1}{\rho} + \frac{\cos\omega}{r}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3} \\ -1, & -1, & \cos\gamma, & \cos\beta \\ -1, & \cos\gamma, & -1, & \cos\alpha \\ -1, & \cos\beta, & \cos\alpha, & -1, \end{vmatrix} = 0.$$

Denoting by Δ the area of the triangle formed by the centres of the given circular arcs, we have (*T.*, § 18)

$$4r^2\Delta^2 = r_1^2 r_2^2 r_3^2 K,$$

whence we deduce, for the radius of the inscribed circle,

$$\begin{aligned}\frac{K}{o} = \frac{8\Delta}{r_1 r_2 r_3}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}&\\ +(1+\cos\alpha)(1-\cos\alpha+\cos\beta+\cos\gamma)\frac{1}{r_1}&\\ +(1+\cos\beta)(1+\cos\alpha-\cos\beta+\cos\gamma)\frac{1}{r_2}&\\ +(1+\cos\gamma)(1+\cos\alpha+\cos\beta-\cos\gamma)\frac{1}{r_3}.&\end{aligned}$$

The Group of Circum-circles, §§ 5–7.

5. From the figure we see that the circum-circle of the triangle ABC cuts the sides at angles

$$\frac{\pi}{2}+(s-\alpha),\quad \frac{\pi}{2}+(s-\beta),\quad \frac{\pi}{2}+(s-\gamma).$$

The radius of the circle is given by (*T.*, § 39)

$$-\frac{2\cos s}{\rho} = \frac{2\Delta}{r_1 r_2 r_3} + \frac{\sin\alpha}{r_1} + \frac{\sin\beta}{r_2} + \frac{\sin\gamma}{r_3},$$

and the angle at which it cuts the orthogonal circle is given by

$$4\cos^2\omega = K\sec^2 s.$$

6. The angles at which the circum-circles of triangle ABC and its associated triangles intersect are easily found from the figure; they are tabulated below:—

	ABC	$A'B'C'$	$A'BC$	$AB'C'$	$AB'C$	$A'BC'$	ABC'	$AB'C$
ABC	π		α	$\beta\sim\gamma$	β	$\gamma\sim\alpha$	γ	$\alpha\sim$
$A'B'C'$		π	$\pi-(\beta\sim\gamma)$	$\pi-\alpha$	$\pi-(\gamma\sim\alpha)$	$\pi-\beta$	$\pi-(\alpha\sim\beta)$	$\pi-\gamma$
$A'BC$	α	$\pi-(\beta\sim\gamma)$	π		$\alpha+\beta-\pi$	$\pi-\gamma$	$\alpha+\gamma-\pi$	$\pi-\beta$
$AB'C'$	$\beta\sim\gamma$	$\pi-\alpha$		π	$\pi-\gamma$	$\alpha+\beta-\pi$	$\pi-\beta$	$\alpha+\gamma-\pi$
$AB'C$	β	$\pi-(\gamma\sim\alpha)$	$\alpha+\beta-\pi$	$\pi-\gamma$	π		$\beta+\gamma-\pi$	$\pi-\alpha$
$A'BC'$	$\gamma\sim\alpha$	$\pi-\beta$	$\pi-\gamma$	$\alpha+\beta-\pi$		π	$\pi-\alpha$	$\beta+\gamma-\pi$
ABC'	γ	$\pi-(\alpha\sim\beta)$	$\alpha+\gamma-\pi$	$\pi-\beta$	$\beta+\gamma-\pi$	$\pi-\alpha$	π	
$A'B'C$	$\alpha\sim\beta$	$\pi-\gamma$	$\pi-\beta$	$\alpha+\gamma-\pi$	$\pi-\alpha$	$\beta+\gamma-\pi$		π

We notice that the circles $AB'C'$, $A'BC'$, $A'B'C$ intersect at angles

$$\beta+\gamma-\pi, \quad \gamma+\alpha-\pi, \quad \alpha+\beta-\pi,$$

and that the circle ABC cuts them at angles

$$\beta \sim \gamma, \quad \gamma \sim \alpha, \quad \alpha \sim \beta,$$

respectively; hence it follows that the circle ABC is the Hart-circle of the triangle $A'B'C'$ formed by the circular arcs $AB'C'$, $A'BC'$, $A'B'C$.

Similarly we see that the circle $A'B'C'$ is the Hart-circle of the triangle ABC formed by the circular arcs $A'BC$, $AB'C$, ABC'.

7. This theorem may be otherwise stated, for we know that the Hart-circle of a triangle touches the inscribed and escribed circles of the triangle; hence—

The circum-circle of any triangle and the circum-circles of the triangles inverse to its associated triangles are touched by four circles, each of which touches one of the former internally and the rest externally.

We thus obtain a group of eight circles, and if we consider that one which touches internally the circum-circle of any triangle, such as ABC, we may say that there is one circle belonging to each of the eight triangles of the system. Thus that circle of the group which corresponds to the triangle ABC will be the circle which touches the circles $AB'C'$, $A'BC'$, $A'B'C$ externally.

In the sequel these circles will be called the T-circles.

The theorem of this article was communicated to me by Mr. A. Larmor, of Clare College, Cambridge, who derived it by considering the dual nature of theorems respecting curves described on a sphere.

The T-circles, §§ 8, 9.

8. To find the angle at which the T-circle corresponding to the triangle ABC cuts the orthogonal circle of the triangle, let 1, 2, 3 denote the given circles, 4 the orthogonal circle, and let 5, 6, 7, 8 denote, respectively, the circles ABC, $AB'C'$, $A'BC'$, $A'B'C$; then, T denoting the circle which touches ABC internally and the others externally, and ω the angle at which the circle T cuts the circle 4, we have, from the formula,

$$\pi \begin{pmatrix} I, 1, 2, 3, 4 \\ 4, 5, 6, 7, 8 \end{pmatrix} = 0,$$

$$\begin{vmatrix} \cos\omega, & 0, & 0, & 0, & -1 \\ -1, & -\sin(s-\alpha), & -\sin(s-\beta), & -\sin(s-\gamma), & \cos\omega_1 \\ 1, & -\sin s, & \sin(s-\gamma), & \sin(s-\beta), & \cos\omega_2 \\ 1, & \sin(s-\gamma), & -\sin s, & \sin(s-\alpha), & \cos\omega_3 \\ 1, & \sin(s-\beta), & \sin(s-\alpha), & -\sin s, & \cos\omega_4 \end{vmatrix} = 0,$$

where

$$-2\cos\omega_1\cos s = 2\cos\omega_2\cos(s-\alpha)$$
$$= 2\cos\omega_3\cos(s-\beta) = 2\cos\omega_4\cos(s-\gamma) = K^2.$$

Hence, we find,

$$K^2\cos\omega = -2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Thus we see that, if the inscribed circle of the triangle ABC cuts the orthogonal circle at the angle Ω,

$$\cos\Omega = 2\cos\omega.$$

Hence, if $\bar{\rho}$, $\bar{\rho}'$ denote the radii of the T-circles corresponding to the triangles ABC, $A'B'C'$, and ρ, ρ' the radii of the inscribed circles of these triangles, we shall have [see $T.$, § 35 (47)]

$$\frac{1}{\rho} - \frac{1}{\rho'} = 2\left(\frac{1}{\bar{\rho}} + \frac{1}{\bar{\rho}'}\right).$$

These two formulæ may be compared with the formulæ obtained in $T.$, § 46, viz.: if the circum-circle and Hart-circle of the triangle ABC cut the orthogonal circle at angles ϕ, Φ, respectively, then

$$\cos\Phi = 2\cos\phi;$$

and if R, R' be the radii of the circum-circles of the triangles ABC, $A'B'C'$, and r, r' the radii of the Hart-circles of these triangles,

$$\frac{1}{r} - \frac{1}{r'} = 2\left(\frac{1}{R} - \frac{1}{R'}\right).$$

9. The radius of the T-circle corresponding to the triangle ABC may be most easily found from the formula

$$\pi\begin{pmatrix} T, & 1, & 2, & 3, & 4 \\ \theta, & 5, & 6, & 7, & 8 \end{pmatrix} = 0,$$

where θ denotes the line at infinity.

Denoting the radius by ρ, we have at once

$$\begin{vmatrix} \frac{1}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \\ -1, & -\sin(s-\alpha), & -\sin(s-\beta), & -\sin(s-\gamma), & \cos\omega_1 \\ 1, & -\sin s, & \sin(s-\gamma), & \sin(s-\beta), & \cos\omega_2 \\ 1, & \sin(s-\gamma), & -\sin s, & \sin(s-\alpha), & \cos\omega_3 \\ 1, & \sin(s-\beta), & \sin(s-\alpha), & -\sin s, & \cos\omega_4 \end{vmatrix} = 0.$$

The Hart-circle, §§ 10–13.

10. The Hart-circle of a triangle might be defined as the circle which cuts the sides at angles $\beta \sim \gamma$, $\gamma \sim \alpha$, $\alpha \sim \beta$; α, β, γ being the angles of the triangle.

If ρ be the radius of the circle, ω the angle at which it cuts the orthogonal circle, we have (*Q.*, § 32)

$$\frac{\cos s}{\rho} = \frac{2\Delta}{r_1 r_2 r_3} + \frac{\cos(s-\alpha)}{r_1} + \frac{\cos(s-\beta)}{r_2} + \frac{\cos(s-\gamma}{r_3},$$

and

$$\cos^2\omega = K \sec^2 s.$$

Starting with this definition we might prove, as in *Q.*, § 32, that the circle touches the inscribed circle and the three escribed circles of the triangle.

11. Several interesting properties of this circle may be derived from considering that the circle ABC is the Hart-circle of the triangle $A'B'C'$ formed by the circular arcs $AB'C'$, $A'BC'$, $A'B'C$.

Substituting the letters P, Q, R for A, B, C, and A, B, C for A', B', C', in §§ 6, 7, we infer that the Hart-circle of the triangle ABC cuts the sides in three points P, Q, R, such that—

(i.) the straight lines AP, BQ, CR meet in a point L;

(ii.) each of the three groups of points (B, C, Q, R), (C, A, R, P), (A, B, P, Q) are concyclic;

(iii.) L is the radical centre of the three circles $BCQR$, $CARP$, $ABPQ$;

(iv.) the Hart-circle PQR is the inverse of the circum-circle ABC with respect to the circle which cuts the circles $BCQR$, $CARP$, $ABPQ$ orthogonally;

(v.) the inverse circles with respect to this orthogonal circle of the arcs BC, CA, AB are the circles circumscribing the triangles AQR, BRP, CPQ.

12. If the circles $BCQR$, $CARP$, $ABPQ$ intersect at angles θ, ϕ, ψ, then, referring to the table given in § 6, we have $\alpha = \phi+\psi-\pi$, &c. Now the circle $BCQR$ cuts the circle BPC at the angle $\frac{3\pi}{2}-\sigma$, where $2\sigma = \theta+\phi+\psi$; hence the circle $BCQR$ cuts the circle BPC at the angle $\frac{3\pi}{4}-\frac{s}{2}$, where $2s = \alpha+\beta+\gamma$.

Hence, the triangle ABC being given, we have the following simple construction for finding the points P, Q, R; draw the circles $BCQR$, $CARP$, $ABPQ$ cutting the arcs BC, CA, AB, respectively, at the angle $\frac{3\pi}{4}-\frac{s}{2}$; and then the circle PQR is the Hart-circle of the triangle ABC.

13. The points P, Q, R evidently correspond to the feet of the perpendiculars of a plane rectilinear triangle, and the point L to the orthocentre. The circle which cuts the circles $BCQR$, $CARP$, $ABPQ$ orthogonally, evidently corresponds to the self-polar circle of a plane rectilinear triangle. Calling this circle the I-circle of the triangle, we see that there is such a circle corresponding to each of the eight triangles.

The I-circle, §§ 14–17.

14. Let θ, ϕ, ψ be the angles of intersection of the circles $BCQR$, $CARP$, $ABPQ$; then, $\omega_1, \omega_2, \omega_3$ being the angles at which the I-circle cuts the circular arcs BCP, CAQ, ABR, we have, by § 5,

$$4\cos^2\omega_1 = L\sec^2(\sigma-\theta),$$

and two similar formulæ, where

$$2\sigma = \theta+\phi+\psi,$$

and $$L = 4\cos\sigma\cos(\sigma-\theta)\cos(\sigma-\phi)\cos(\sigma-\psi).$$

But $$\alpha = \phi+\psi-\pi, \quad \beta = \psi+\theta-\pi, \quad \gamma = \theta+\phi-\pi;$$

therefore $$s = 2\sigma-\frac{3\pi}{2},$$

and $$L = 4\cos\left(\frac{s}{2}+\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{4}+\alpha-\frac{s}{2}\right)$$
$$\cos\left(\frac{\pi}{4}+\beta-\frac{s}{2}\right)\cos\left(\frac{\pi}{4}+\gamma-\frac{s}{2}\right).$$

Hence

$$\frac{\cos^2\omega_1}{\sec^2\left(\frac{\pi}{4}+\alpha-\frac{s}{2}\right)} = \frac{\cos^2\omega_2}{\sec^2\left(\frac{\pi}{4}+\beta-\frac{s}{2}\right)} = \frac{\cos^2\omega_3}{\sec^2\left(\frac{\pi}{4}+\gamma-\frac{s}{2}\right)} = \tfrac{1}{4}L.$$

15. Again, let S, H denote the circum-circle and Hart-circle of the triangle; then, X denoting any other circle, we have, since S is the inverse of H with respect to I,

$$\frac{\pi_{X,S}}{\pi_{S,I}} + \frac{\pi_{H,X}}{\pi_{I,H}} = 2\frac{\pi_{X,I}}{\pi_{I,I}};$$

whence, taking X to be the line at infinity, we have

$$\frac{1}{\pi_{S,I}} + \frac{1}{\pi_{I,H}} = \frac{2}{\pi_{I,I}};$$

or, if R, r are the radii of S and H, and ρ the radius of I, and ω the angle at which S cuts I,

$$\frac{1}{R} - \frac{1}{r} = \frac{2\cos\omega}{\rho};$$

and we know, by § 5, that

$$4\cos^2\omega = L\sec^2\sigma,$$

where L, σ have the same values as in § 14.

Hence, by § 5 and § 10,

$$\frac{L^{\frac{1}{2}}\sec\left(\frac{s}{2}+\frac{3\pi}{4}\right)}{\rho} = \frac{1}{R} \sim \frac{1}{\rho},$$

or
$$\frac{4L^{\frac{1}{2}}\cos\left(\frac{s}{2}+\frac{\pi}{4}\right)}{\rho} = \frac{2\cos s}{R} \sim \frac{2\cos s}{\rho}$$
$$= \frac{2\Delta}{r_1r_2r_3} + \frac{1}{r_1}\left[2\cos(s-\alpha)-\sin\alpha\right]$$
$$+ \frac{1}{r_2}\left[2\cos(s-\beta)-\sin\beta\right]$$
$$+ \frac{1}{r_3}\left[2\cos(s-\gamma)-\sin\gamma\right].$$

16. To find the angle χ at which the I-circle cuts the orthogonal circle, let ϖ and ϖ' be the angles at which the circles S, H cut the orthogonal circle, and then, taking X as this circle, the formula given in § 15 reduces to

$$\cos\varpi \sim \cos\varpi' = \cos\chi\cos\omega.$$

But $\cos^2\varpi' = K\sec^2 s$, by § 10,

and $4\cos^2\varpi = K\sec^2 s$, by § 5;

hence $K^{\frac{1}{2}}\sec s = 2\cos\chi\, L^{\frac{1}{2}}\sec\sigma$,

therefore
$$\cos^2\chi = \frac{1}{4}\frac{K}{L}\frac{\cos^2\sigma}{\cos^2 s}$$

$$= \frac{1}{4}\frac{\cos^2\left(\frac{3\pi}{4}+\frac{s}{2}\right)}{\cos^2 s}\cdot\frac{K}{L}$$

$$= \frac{1}{16}\sec^2\left(\frac{\pi}{4}+\frac{s}{2}\right)\cdot\frac{K}{L}.$$

17. We may notice that the I-circle of the triangle ABC is real when L is positive; this can only be the case when one of the angles,

$$\left(\frac{\pi}{4}+\alpha-\frac{s}{2}\right) \text{ say, is } > \frac{\pi}{2};$$

i.e., provided that one of the angles of the triangle ABC, α is greater than $\frac{1}{2}(\beta+\gamma+\pi)$.

Thursday, May 8th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

The President announced that a member, Lieut.-Col. J. R. Campbell, had asked to be allowed to give a donation of £500 to the Society, and that the Council had accepted the gift. On the motion of the Treasurer, seconded by Mr. S. Roberts, the following resolution was carried unanimously:—"That the cordial thanks of the London Mathematical Society be given to Lieut.-Col. Campbell for his generous gift of £500 to the general fund of the Society."

The following communications were made:—

On the Function which denotes the Excess of the Divisors of a Number which $\equiv 1$, mod. 3, over those of a Number which $\equiv 2$, mod. 3: Dr. Glaisher.

A Table of Complex Multiplication Moduli: Prof. Greenhill.

On a Theorem relating to Bicircular Quartics and Twisted Quartics: R. Lachlan.

On the Genesis of Binodal Quartic Curves from Conics: H. M. Jeffery.

On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$: Prof. Mathews.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVII., No. 288.

"Educational Times," for May.

"Mathematics from the 'Educational Times,'" Vol. LII.

"Transactions of the Royal Irish Academy," Vol. XXIX., Part XIII.

"Annals of Mathematics," Vol. V., No. 2; University of Virginia, Oct., 1889.

"Bulletin des Sciences Mathématiques," Tome XIV., April, 1890.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 6; Tome XVIII., Nos. 2, 3, and 4.

"Atti della Reale Accademia dei Lincei," Vol. VI., Fasc. 4 and 5; Roma, 1890.

"Journal de l'École Polytechnique," 59th Cahier; Paris, 1889.

"Annuario del Circolo Matematico di Palermo—Rules," &c.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 102, 103, and 104.

"Jahrbuch über die Fortschritte der Mathematik," Band XIX., Heft 2; Jahrgang, 1887.

"Journal für die Reine und Angewandte Mathematik," Band CVI., Heft 2.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome IV.; Tome V., 1er Cahier.

"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde de Juin 1887 à Mai 1888, et de Juin 1888 à Mai 1889; Bordeaux.

"Esquisse Historique sur la Marche du Développement de la Géométrie du Triangle," by E. Vigarié; 8vo pamphlet (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Axes de Steiner et l'Hyperbole de Kiepert," et "Sur les Foyers de Steiner d'un Triangle," MM. J. Neuberg and A. Gob (read before the French Association for the Advancement of Science—Congrès de Paris—1889).

"Sur les Projections et Contre-Projections d'un Triangle Fixe, et Sur le Système de Trois Figures directement Semblables," M. J. Neuberg. (Offprint from Tome XLIV. of the "Mémoires Couronnés et autres Mémoires," publiés par l'Académie Royale de Belgique, 1890.)

On a Theorem relating to Bicircular Quartics and Twisted Quartics. *By* R. LACHLAN, M.A.

[*Read May 8th*, 1890.]

The present paper deals with a theorem analogous to the theorem that: Every plane cubic which passes through eight fixed points must pass through a ninth. It will be proved that a very similar theorem exists for plane bicircular quartics, and for twisted quartic curves, and, as might be expected, the theorem leads to as many, if not more, interesting results than the one for the cubic. It has been thought advisable to state for circular curves the corresponding theorems to those which are given in Salmon ("Higher Plane Curves," §§ 30–34) for ordinary plane curves, although of course the former may be easily deduced from these. And it should be stated that a circular curve of the $2n^{\text{th}}$ order is always understood to mean a curve of order $2n$, having each of the circular points at infinity as multiple points of the n^{th} order.

The paper is divided into three sections: in the first are given the general theory of the intersection of circular curves; in the second section a particular theorem relating to plane bicircular quartics is developed; and in the third section the corresponding theorem for twisted quartics is proved independently, and it is explained how the particular theorems in the second section may be translated so as to apply to twisted quartics.

General Theorems on Circular Curves, §§ 1–4.

1. The general equation of a curve of the $2n^{\text{th}}$ order having multiple points of the n^{th} order at each of the circular points, is of

the form

$$u_0(x^2+y^2)^n+u_1(x^2+y^2)^{n-1}+\ldots+u_{n-1}(x^2+y^2)$$
$$+v_n+v_{n-1}+\ldots v_1+v_0=0,$$

where u_r and v_r are homogeneous expressions of the r^{th} degree in x and y.

Such a curve is determined when we have given

$$(1+2+\ldots+n+n+1+n+\ldots+1)-1,$$

i.e., $$n(n+2) \text{ points.}$$

2. Two circular curves of the $2m^{\text{th}}$ and $2n^{\text{th}}$ orders intersect in $2mn$ points, excluding the circular points. Hence it follows that every circular curve of the $2n^{\text{th}}$ degree which passes through $n(n+2)-1$ fixed points must also pass through $(n-1)^2$ other fixed points.

We may also deduce the theorem: If of the $2n^2$ points of intersection of two circular curves each of the $2n^{\text{th}}$ order, $2np$ lie on a circular curve of order $2p$, the remaining $2n(n-p)$ will lie on a circular curve of order $2(n-p)$.

3. Again, every circular curve of the $2n^{\text{th}}$ order which passes through $2np-(p-1)^2$ points on a circular curve of order $2p$ (p being less than n) meets this curve in $(p-1)^2$ other fixed points.

For through any other $(n-p)(n-p+2)$ points on the curve of order $2n$ we can draw a circular curve of order $2(n-p)$, and this, with the curve of order $2p$, makes up a curve of order $2n$ passing through

$$2np-(p-1)^2+(n-p)(n-p+2),$$

i.e., $n(n+2)-1$ points; hence, by § 2, it must pass through $(n-1)^2$ other fixed points on the given curve of order $2n$; and these must lie on one or other of the curves of orders $2p$ and $2(n-p)$; but these curves can only meet the given curve in $2np$ and $2n(n-p)$ points, respectively; hence the truth of the theorem is manifest.

As a particular case, we see that every circular curve of the eighth order which passes through 15 points on a bicircular quartic passes through one other fixed point; and every circular curve of the order $2n$ which passes through $4n-1$ fixed points on a bicircular quartic must pass through one other fixed point.

4. An extension of the theorem in § 3 may be thus stated:—

Any circular curve of the $2r^{\text{th}}$ order (r being greater than m and n,

but less than $m+n-1$), which passes through all but $(m+n-r-1)^2$ of the $2mn$ points of intersection of two circular curves of orders $2m$ and $2n$, will pass also through the remaining intersections.

For, if we draw a circular curve of order $2(r-m)$ through

$$(r-m)(r-m-2)$$

arbitrary points on the curve of order $2n$; and a circular curve of order $2(r-n)$ through $(r-n)(r-n-2)$ arbitrary points on the curve of order $2m$; these curves make up, with the curves of orders $2m$ and $2n$, two curves each of order $2r$, and each passing through such a number of the $2mn$ points of intersection of the given curves of orders $2m$, $2n$ as make up $r(r+2)-1$ points in each case; and hence these two curves pass through $(r-1)^2$ other fixed points.

But

$$r(r+2)-1-(r-m)(r-m+2)-(r-n)(r-n-2)$$

$$= 2mn-(m+n-r-1)^2;$$

hence the theorem is seen to be true. For the theorem to be applicable, it is easily seen that r must be at least equal to the greater of the integers m and n, also $r-m$ must be $< n$, otherwise it would not be possible to draw the auxiliary curve of order $2(r-m)$; and, as the theorem is nugatory when $r = m+n-1$, we see that we must have

$$r < m+n-1.$$

Application to Bicircular Quartics, §§ 5–12.

5. The general theorem in § 3 states that every circular curve of the eighth degree which passes through 15 fixed points on a bicircular quartic must pass through a sixteenth fixed point; in other words, if two circular systems each of the eighth degree intersect in 15 points which lie on a bicircular quartic, their sixteenth point of intersection also lies on the quartic.

Suppose we have three circles A, B, C cutting a bicircular quartic respectively in the points a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; and let the circles (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) meet the quartic in the points d_1, d_2, d_3, d_4, respectively; then these points must be concyclic. For let the circles (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4) be denoted by L, M, N, P; and let D denote the circle (d_1, d_2, d_3); then each of the systems A, B, C, D and L, M, N, P are of the eighth degree, and fifteen of their points of intersection lie on the quartic, therefore the remaining point in which the system A, B,

C, D meets the quartic must be the point d_4, which is the sixteenth point in which the system L, M, N, P meets the quartic. Hence the circle D must pass through the point d_4.

6. Suppose now that the circles A, B, C coincide; then the circles L, M, N, P become the osculating circles at the points a_1, a_2, a_3, a_4; hence we have the theorem that: The osculating circles at four concyclic points on a bicircular quartic, meet the quartic in four points which are also concyclic.

The circle D passing through the points d_1, d_2, d_3, d_4, in which the osculating circles at a_1, a_2, a_3, a_4 meet the quartic, might be called the *satellite* circle of the circle A.

7. Let us now suppose the points a_1, a_2, a_3 are cyclic points; then the circle D must coincide with the circle A; and hence the point a_4 must either be another cyclic point, or must coincide with one of the points a_1, a_2, a_3; in other words, the circle which can be drawn passing through any three cyclic points, meets the quartic in a fourth cyclic point or touches it at one of the three points.

Now we know that there are sixteen cyclic points on a bicircular quartic, viz., four cyclic points on each of the principal circles of the quartic.

Let a_1, a_2, a_3, a_4 be the four cyclic points on the principal circle J_1; β_1, β_2, β_3, β_4 the cyclic points on J_2; γ_1, γ_2, γ_3, γ_4 those on J_3; and δ_1, δ_2, δ_3, δ_4 those on J_4. If a_2 be the inverse point to a_1 with respect to J_2, it is evident that the circles $a_1a_2\beta_1$, $a_1a_2\beta_2$, $a_1a_2\beta_3$, $a_1a_2\beta_4$ will touch the quartic at the points β_1, β_2, β_3, β_4, respectively. And it follows from the theorem that two pairs of inverse points are concyclic, that no other circle passing through a_1, a_2 and a third cyclic point, γ_1 say, touch the curve at either of the points a_1, a_2, γ_1. Now sixteen points may be arranged in groups of three in $16 \cdot 15 \cdot 14/6 = 560$ ways; through each pair of points such as a_1, a_2 we can draw four circles touching the curve at another cyclic point, and we can take a pair such as a_1, a_2 in 6×4 ways (viz., 6 pairs on each principal circle); hence, of the 560 circles that are obtained by taking any three of the 16 points, $6 \times 4 \times 4 = 96$ touch the quartic; we have left 464 circles, and, by what was proved above, each of these must meet the curve in a fourth cyclic point. Dividing this number by 4, we see that 116 circles can be drawn each passing through four cyclic points on a bicircular quartic.

It will be interesting to see how these circles may be grouped. There are the four principal circles; and then we know that pairs of inverse points are concyclic, thus we shall have 48 circles of the type

$a_1a_2\gamma_1\gamma_2$, a_1 and a_2 being inverse points with respect to J_2, as also γ_1 and γ_2: we shall also have the circles $a_1a_2\gamma_3\gamma_4$, $a_1a_2\delta_1\delta_2$, $a_1a_2\delta_3\delta_4$; *i.e.*, four circles for each pair of points a_1a_2 on J_1, and so $\frac{1}{2}.4\times6\times4 = 48$ in all; the remaining 64 circles will be circles passing through one point on each of the principal circles.

8. We know that nine circles can always be drawn through any point a on a bicircular quartic which will osculate the curve elsewhere; let us suppose, then, that the osculating circles at the points $a_1, a_2, \ldots a_9$ meet the curve again in the point a; and let the osculating circles at $b_1, b_2, \ldots b_9$ meet the curve in b, the osculating circles at $c_1, c_2, \ldots c_9$ meet in c, and the osculating circles at $d_1, d_2, \ldots d_9$ meet the curve in d. Then, if the points a, b, c, d are concyclic, it follows from § 6 that the circle passing through three points such as a_i, b_j, c_k must pass through one of the points $d_1, d_2, \ldots d_9$.

Consequently, though there corresponds but one satellite circle D for every circle A, yet corresponding to a circle D there are $9\times9\times9 = 729$ different circles A.

9. Let us suppose that the points a, c, d coincide; then we see that, if the osculating circle at the point a meets the curve in the point b, and if the osculating circles at $a_1, a_2, \ldots a_9$ meet the curve in the point a, and the osculating circles at $b_1, b_2, \ldots b_8$ meet the curve in b, then the circle passing through any three of the points $a_1, a_2, \ldots a_9$ must pass through one of the points $b_1, b_2, \ldots b_8$. In other words, if $a_1, a_2, \ldots a_9$ are the points, the osculating circles at which meet the curve in the point a, then the $9.8.7/6 = 84$ circles which can be drawn through the points $a_1, a_2, \ldots a_9$, taken three at a time, meet again in eight other points which are also on the quartic, and the osculating circles at these eight points meet the curve again in the same point as the osculating circle at the point a.

10. Returning to the general theorem in § 5, let us suppose that the circles A, B, C are bitangent circles to the quartic at the points a, a'; b, b'; c, c', respectively; then the circles L, M coincide, and also the circles N, P. Hence, if the circles abc, $a'b'c'$ cut the curve again in the points d, d', we infer that the circle which touches the curve at d and passes through d' must touch the curve at d'. If the three bitangent circles aa', bb', cc' belong to the same system, *i.e.*, cut orthogonally the same principal circle, then it is obvious that d, d' are inverse points with respect to the same principal circle; but in the above reasoning it would seem that the three bitangent circles A, B, C need not necessarily belong to the same system.

11. Further, let us suppose that the points a, b, c coincide; then we see that, if three of the bitangent circles at the point a touch the curve again at the points a_1, a_2, a_3, respectively, and if the osculating circle at a and the circle $a_1a_2a_3$ meet the curve again in d, d', then d, d' are points of contact of a bitangent circle.

Now there are four circles which touch at a and touch the curve elsewhere; let a_1, a_2, a_3, a_4 be the points of contact (*i.e.*, the inverse points of a with respect to the four principal circles), then we infer that the four circles which can be drawn through these four points meet the curve in four points d_1, d_2, d_3, d_4 which are the points of contact of the bitangent circles which touch the curve at d, the point where the osculating circle at a cuts the curve.

12. Again, let the points of contact of the bitangent circles at a, b, c, d be respectively a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4; then, if a, b, c, d are concyclic, we see, from § 10, that any circle such as $a_ib_jc_k$ must pass through one of the four points d_1, d_2, d_3, d_4.

Hence these 16 points lie on 64 circles, each circle passing through one and only one of the points corresponding to a, b, c, d, respectively.

Analogous Theorems in connexion with Twisted Quartics, §§ 13–15.

13. Since the equation of a bicircular quartic may be written in the form

$$ax^2+by^2+cz^2+dw^2=0,$$

where x, y, z, w are the "power-coordinates" of a point referred to four mutually orthogonal circles, so that they are connected by the identical relation

$$x^2+y^2+z^2+w^2=0,$$

it follows that, corresponding to all theorems connected with bicircular quartics, there exist analogous theorems concerning quartic curves in space. In fact, we have merely to substitute the word "plane" for "circle" in the enunciation of theorems concerning bicircular quartics to obtain the corresponding theorem for twisted quartics.

Thus, in place of the theorem in § 5, we shall have the theorem: Every quartic surface which passes through 15 fixed points on the curve of intersection of two quadrics must pass through one other fixed point on the curve.

14. It may be as well to give an independent proof of this theorem.

Thus, let u, v denote two quadrics, and S any quartic surface passing through 15 fixed points on the curve of intersection of u and v; we have to prove that any other quartic surface which passes through these 15 points must pass through the remaining point of intersection of S with u and v.

Now, through any 8 arbitrary points on the curve of intersection of u and S, and through an arbitrary point P on S, we can draw a quadric, v' say; also, through 8 arbitrary points on the curve of intersection of S and v, we can draw a quadric, u' say, also passing through P (P being supposed not to lie on either u or v); then we have three quartic surfaces, S, uu', vv', each passing through $15+8+8+1=32$ fixed points, and every quartic which passes through these points must be of the form $S+\lambda uu'+\mu vv'$, and therefore must pass through the remaining 32 points in which S, uu', and vv' intersect.

Hence every quartic surface which passes through 15 of the points of intersection of S, u, v must pass through the remaining point of intersection.

15. We have at once the theorem, that if three planes A, B, C be drawn cutting a twisted quartic in the points a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, respectively; the planes $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_4b_4c_4$ will cut the quartic in four coplanar points d_1, d_2, d_3, d_4.

Consequently, all the theorems stated previously may be at once translated so as to apply to twisted quartics.

On the Arithmetical Theory of the Form $x^3+ny^3+n^2z^3-3nxyz$.

By Professor MATHEWS, M.A.

[*Read May 8th*, 1890.]

In the last four papers contained in Vol. I. of Dirichlet's collected works will be found some remarkable propositions relating to certain arithmetical forms of higher degrees.*

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* Dirichlet's *Werke*, I., pp. 619, 625, 633, 639. The titles of the papers are—"Sur la Théorie des Nombres" (*Comptes Rendus*, 1840, p. 285, or *Liouville*, Sér. I., t. v., p. 72); "Einige Resultate von Untersuchungen über eine Classe homogener Functionen des dritten und der höheren Grade" (*Berichte über die Verhandlungen d. Königl. Preuss. Akad. d. Wissensch.*, 1841, p. 280); "Verallgemeinerung eines

In the following note I propose to illustrate Dirichlet's method by applying it to a special case, which is otherwise of interest in connexion with an algorithm of Jacobi's (*Crelle*, Vol. LXIX., p. 29).

The form to be considered is

$$F(x, y, z) \equiv x^3+ny^3+n^2z^3-3nxyz,$$

and it will be shown—

(i.) that if an integer m can be represented by the form $F(x, y, z)$ at all, it can be so represented in an infinite number of ways;

(ii.) that the Diophantine equation $F(x, y, z) = 1$ can always be satisfied by integral values of x, y, z;

(iii.) that all the integral solutions of $F(x, y, z) = 1$ can be derived from a single fundamental solution in a manner analogous to that in which the solutions of the Pellian equation are obtained.

1. Let n be a positive integer, t the real cube root of n, θ a complex cube root of unity; then

$$\begin{aligned} F(x, y, z) &= (x+yt+zt^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2) \\ &= (x+yt+zt^2)\{(x^2-nyz)+(nz^2-xy)\,t+(y^2-zx)\,t^2\} \\ &= x^3+ny^3+n^2z^3-3nxyz. \end{aligned}$$

Dirichlet has shown (*Werke* I., p. 635) that we can choose, in an infinite number of ways, a set of integers x, y, z, so that

$$|\,x+yt+zt^2\,| < 1/\mu^2,$$

where μ is the greater of the quantities $|y|, |z|$. (Here, as in what follows, $|z|$ denotes the absolute value or modulus of z.)

His proof is as follows:—Choose for y, z all pairs of values, the same or different, which can be obtained from

$$-p, -(p-1), \ldots -1, 0, 1, \ldots (p-1), p$$

where p is any positive integer. Having chosen any pair, let x be determined so that $x+yt+zt^2$ may be less than 1, and not negative;

Satzes aus der Lehre von den Kettenbrüchen ..." (*ibid.*, 1842, p. 93); and "Zur Theorie der Complexen Einheiten" (*ibid.*, 1846, p. 103).

See also Eisenstein's memoir on cubic forms (*Crelle*, t. XXVIII., p. 289, and XXIX., p. 19); Sylvester, "On the General Solution in certain cases of the Equation $x^3+y^3+Az^3 = Mxyz$," &c., *Phil. Mag.*, XXXI. (1847), pp. 461–7, and "On certain Ternary Cubic Form Equations," *Amer. Math. Jour.*, t. III. (1880), pp. 58–88 and 179–189.

For the last two references I am indebted to Professor Cayley.

we thus obtain for the expression one value zero, and $(2p+1)^3-1$ other values which are all positive proper fractions. Now consider the series of terms

$$0, \quad \frac{1}{4p^2}, \quad \frac{2}{4p^2}, \quad \ldots\ldots \quad \frac{4p^2-1}{4p^2}, \quad 1;$$

these divide the interval $0 \ldots 1$ into $4p^2$ equal parts; and, since

$$(2p+1)^3-1 > 4p^2,$$

it follows that, of the proper fractions previously obtained, it will always be possible to find two which are in the same one of the $4p^2$ intervals. Let

$$x_1+y_1t+z_1t^2 \quad \text{and} \quad x_2+y_2t+z_2t^2$$

be two such fractions. They are not equal, for this would imply

$$x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2;$$

hence their difference is less than $1/4p^2$ and not zero; further, the differences (y_1-y_2), (z_1-z_2) are both numerically not greater than $2p$; hence, if we put

$$x = x_1-x_2, \quad y = y_1-y_2, \quad z = z_1-z_2,$$

$$|x+yt+zt^2| < 1/4p^2,$$

and therefore *a fortiori* $< 1/\mu^2$ if μ is the greater of the quantities $|y|$, $|z|$.

Since p was chosen at pleasure, it follows that an infinite number of integers x, y, z can be chosen in the manner stated.

If (x, y, z) be any such set,

$$\begin{aligned} |x+y\theta t+z\theta^2t^2| &= |x+yt+zt^2+(\theta-1)yt+(\theta^2-1)zt^2| \\ &< |x+yt+zt^2| + |(\theta-1)yt + (\theta^2-1)zt^2| \\ &< |x+yt+zt^2| + \{|yt| + |zt^2|\}\sqrt{3} \\ &\qquad \text{(since } |\theta-1| = |\theta^2-1| = \sqrt{3}\text{)}, \end{aligned}$$

and *a fortiori* $$< \frac{1}{\mu^2} + (t^2+t)\mu\sqrt{3}.$$

Similarly, $$|x+y\theta^2t+z\theta t^2| < \frac{1}{\mu^2} + (t^2+t)\mu\sqrt{3};$$

and therefore $|F(x,y,z)| < \frac{1}{\mu^3}\{(t^2+t)\mu\sqrt{3}+1/\mu^2\}^2$

$$< 3(t^2+t)^2+2\sqrt{3}(t^2+t)/\mu^3+1/\mu^6.$$

Since μ is not less than 1, it follows *a fortiori* that

$$|F(x,y,z)| < \{\sqrt{3}(t^2+t)+1\}^2.$$

If, then, M is the integer next greater than

$$\{\sqrt{3}(t^2+t)+1\}^2,$$

there are an infinite number of sets of integers x, y, z such that

$$|F(x,y,z)| < M.$$

But $F(x,y,z)$ is always an integer when x, y, z are integral; therefore at least one value of F must correspond to an infinite number of sets (x,y,z): that is, there must be at least one integer m such that the Diophantine equation $F(x,y,z)=m$ admits of an infinity of integral solutions.

2. Let (x,y,z), (x',y',z') be any two sets such that

$$F(x',y',z') = F(x,y,z) = m.$$

The expression

$$\frac{x'+y't+z't^2}{x+yt+zt^2} = \frac{(x'+y't+z't^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)}{F(x,y,z)}$$

$$= \frac{X+Yt+Zt^2}{m},$$

where X, Y, Z are integers, the reduction in the numerator being made by help of $t^3=n$.

Now we may suppose (x',y',z') congruous (mod. m) to (x,y,z): that is to say,

$$x' \equiv x, \quad y' \equiv y, \quad z' \equiv z \pmod{m};$$

for not more than m^3 sets can be incongruous, and there are ∞ sets.

This being so, we have

$$(X-m)+Yt+Zt^2 = \{(x'-x)+(y'-y)t+(z'-z)t^2\}$$

$$(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)$$

$$= m(\alpha+\beta t+\gamma t^2)(x+y\theta t+z\theta^2t^2)(x+y\theta^2t+z\theta t^2)$$

$$= m(A+Bt+Ct^2)$$

where A, B, C are integers.

Hence $$X \equiv Y \equiv Z \equiv 0 \pmod{m}.$$

so that $$\frac{x'+y't+z't^2}{x+yt+zt^2} = \xi+\eta t+\zeta t^2$$

where ξ, η, ζ are integers; and therefore

$$F(\xi, \eta, \zeta) = \frac{F(x', y', z')}{F(x, y, z)} = 1;$$

that is, (ξ, η, ζ) is an integral solution of

$$F(x, y, z) = 1.$$

3. It is clear that, if (ξ, η, ζ) is any solution, and we put

$$(\xi+\eta t+\zeta t^2)^k = \xi_k+\eta_k t+\zeta_k t^2$$

where k is any integer, then (ξ_k, η_k, ζ_k) will also be a solution. It will now be proved that every integral solution can be derived in this way from a certain fundamental solution.

Write $$u = x+yt+zt^2,$$

$$v = x+y\theta t+z\theta^2 t^2,$$

$$w = x+y\theta^2 t+z\theta t^2,$$

so that, if x, y, z are taken to be ordinary rectangular coordinates, $u = 0$ represents a real plane, and $v = 0$, $w = 0$ two conjugate imaginary planes intersecting in a real line. The surface

$$F(x, y, z)-1 = 0$$

has a real asymptotic plane $u = 0$ and a real asymptotic line ($v = 0$, $w = 0$).

It may be conceived to be generated by the intersection of the plane

$$u = c$$

with the elliptic cylinder $$vw = \frac{1}{c},$$

c being a variable parameter.

For real points on the surface vw, and therefore c, is essentially positive, so that the real part of the surface consists of a single sheet on one side of $u = 0$ with a funnel-shaped depression extending to infinity in the direction of the line ($v = 0$, $w = 0$).

If x, y, z are whole numbers, let (x, y, z) be called an integral point; and if, moreover,

$$F(x, y, z) = 1,$$

let it be called a radical point.

The plane $$w-1 = 0$$

goes through the radical point (1, 0, 0), and cuts the surface in an ellipse. Let the plane be moved parallel to itself until it *first* passes through another radical point (ξ_0, η_0, ζ_0). This point is not infinitely near (1, 0, 0), and (on account of the shape of the surface) it is not at an infinite distance from the origin. Hence the plane will have moved through a finite distance; *i.e.*,

$$1 \ldots (\xi_0+\eta_0 t+\zeta_0 t^2)$$

will be a finite interval, and there will be no radical point (ξ, η, ζ) for which

$$1 < \xi+\eta t+\zeta t^2 < \xi_0+\eta_0 t+\zeta_0 t^2$$

(where the signs of inequality must be reversed if $\xi_0+\eta_0 t+\zeta_0 t^2 < 1$).

Now, let (x, y, z) be any other radical point; then, if h is any integer, and we put

$$\frac{(x+yt+zt^2)}{(\xi_0+\eta_0 t+\zeta_0 t^2)^h} = \alpha+\beta t+\gamma t^2,$$

then (α, β, γ) is also a radical point.

Denoting by $\log(x+yt+zt^2)$ the real logarithm of $x+yt+zt^2$, we have

$$\log(\alpha+\beta t+\gamma t^2) = \log(x+yt+zt^2) - h\log(\xi_0+\eta_0 t+\zeta_0 t^2).$$

If $\log(x+yt+zt^2)$ is not a multiple of $\log(\xi_0+\eta_0 t+\zeta_0 t^2)$, it will be possible to determine h so that the expression on the right-hand is intermediate between 0 and $\log(\xi_0+\eta_0 t+\zeta_0 t^2)$, and therefore $\alpha+\beta t+\gamma t^2$ intermediate between 1 and $\xi_0+\eta_0 t+\zeta_0 t^2$: but this has been shown to be impossible; therefore

$$x+yt+zt^2 = (\xi_0+\eta_0 t+\zeta_0 t^2)^k$$

where k is some positive or negative integer.

If $$\frac{1}{\xi_0+\eta_0 t+\zeta_0 t} = \xi_0'+\eta_0' t+\zeta_0' t,$$

then $(\xi_0', \eta_0', \zeta_0')$ has just the same right as (ξ_0, η_0, ζ_0) to be called a fundamental solution; in the geometrical theory we see, correspond-

ingly, that the plane $u-1=0$ may be moved in either direction in order to come to a fundamental point.

4. Various consequences of the preceding theory are immediately evident. Thus all numbers representable by F are cubic residues of n; the product of any number of such integers is representable by F; and if $F(x, y, z)$ be any representation of m, then all the representations are found from

$$x'+y't+z't^2 = (x+yt+zt^2)(\xi_0+\eta_0 t+\zeta_0 t^2)^k.$$

Moreover, this relation between (x', y', z') and (x, y, z) gives an infinity of linear transformations of the form F into itself, the coefficients being integral, and their determinant equal to 1.

The direction cosines of the asymptotic line of the surface

$$F(x, y, z) = m$$

are given by $$\lambda : \mu : \nu = t^2 : t : 1;$$

from which it may be inferred that, among the solutions of

$$F(x, y, z) = m,$$

there are an infinite number for which x, y, z are all positive.

If we take a solution (ξ, η, ζ) of

$$F(x, y, z) = 1,$$

for which $$\xi+\eta t+\zeta t^2 > 1,$$

and put, as before, $$(\xi+\eta t+\zeta t^2)^k = \xi_k+\eta_k t+\zeta_k t^2,$$

k being a positive integer, then, as k increases indefinitely, we have at last

$$\xi_k : \eta_k : \zeta_k = t^2 : t : 1,$$

that is, $$\xi_k/\zeta_k, \quad \eta_k/\zeta_k$$

are approximations, more and more accurate as k increases, to t^2 and t respectively. By successive squaring, the approximation may be made with considerable rapidity.

For example, if $n = 5$, we may put

$$\xi = 41, \quad \eta = 24, \quad \zeta = 14,$$

and hence $$\xi_2 = 5041, \quad \eta_2 = 2948, \quad \zeta_2 = 1724,$$

and so on; where, in general,

$$\xi_{2k} = \xi_k^2+10\eta_k\zeta_k,$$

$$\eta_{2k} = 5\zeta_k^2+2\xi_k\eta_k,$$

$$\zeta_{2k} = \eta_k^2+2\zeta_k\xi_k.$$

It will be found that $\eta_3/\zeta_3 = 1{\cdot}709976\ldots,$

the true value of $\sqrt[3]{5}$ being $1{\cdot}7099759\ldots.$

For some of the simplest values of n, I have calculated the following solutions by Jacobi's method; I believe they are fundamental, but this remains to be proved.

n	x,	y,	z
2	1,	1,	1
3	4,	3,	2
4	5,	3,	2
5	41,	24,	14
7	4,	2,	1
11	89,	40,	18

On the Genesis of Binodal Quartic Curves from Conics.

By Henry M. Jeffery, F.R.S.

[*Read May 8th*, 1890.]

1. In this memoir I propose to generalise Laguerre's method of generating bicircular quartics for two distinct families of binodal quartics—(1) those which pass through two assigned points, (2) those which touch a given conic in four assigned points. The processes used in (1) and (2) are adapted from those employed for plane bicircular quartics and spherical sphero-quadrics respectively.

In a second memoir I hope to explain the classification of binodal quartics.

I. *On Binodal Quartics which are to be drawn through two assigned points.*

2. DEF.—Two conics P, Q intersect a third S in the same two points, and therefore all three P, Q, S also intersect in three concurrent chords; anorthotomic intersection of P, Q occurs, when the pole of the common chord of P, Q, S with respect to P or Q is also the pole of the second chord of intersection of P and Q with respect to Q or P respectively.

This may be better understood by considering the special case of orthotomic intersection.

Two circles P, Q, which pass through the same two points at infinity, intersect orthogonally, when the pole of the line at infinity with respect to P or Q is also the pole of their common chord with respect to Q or P respectively.

The definition is founded on the following theorem:—

"Two conics P, Q, intersecting in four points A, B, C, D, may be such that the pole of AB in regard to Q = the pole of CD in regard to P; and when this is so, then also the pole of AB in regard to P = the pole of CD in regard to Q."

The third conic S is introduced for analytical convenience only.

The term "anorthotomic" is used in a definite sense, both here and in Part II. (where it differs, but is also definite), and is not the mere negative of "orthotomic." It has been coined after the analogy of "anharmonic," by which Chasles denoted a definite ratio.

"Prof. Cayley would rather use 'orthotomic,' explaining the generalised sense in which the word is to be taken. And so in other like cases, he would retain the words 'circle,' 'centre,' &c., explaining by way of definition the generalised senses in which these words are to be taken, viz., considering any two points I, J, a conic through these two points is said to be a circle, and the polar of IJ in regard to the curve is said to be the centre, &c. Two circles intersect in the points I, J, and besides in two other points; the line joining these is said to be the common chord. And then, for orthotomic circles, two circles may be such that the centre of the first is the pole of the common chord with respect to the second of them, and, when this is so, then also the centre of the second is the pole of the common chord with respect to the first of them; two circles thus related to each other are said to be orthotomic," &c., &c.

The readers of this memoir may substitute "circles" for "conchordal conics," which has been retained in the text as a distinctive term.

Bicuspidal (§ 14) and tacnodal (§ 17) quartics are generated from conics as a particular case and the limiting form respectively of binodal quartics.

3. To find the condition of anorthotomic intersection for the two conchordal conics

$$S+(l\alpha+m\beta+n\gamma)(d\alpha+e\beta+f\gamma)=0 \quad \ldots\ldots\ldots\ldots\ldots (P),$$

$$S+(l\alpha+m\beta+n\gamma)(g\alpha+h\beta+k\gamma)=0 \quad \ldots\ldots\ldots\ldots\ldots (Q),$$

$$S\equiv (u, v, w, u', v', w')(\alpha, \beta, \gamma)^2.$$

The polar of a point (x, y, z) with respect to (P) will be the common chord

$$\left[x(d-g)+y(e-h)+z(f-k)\right](l\alpha+m\beta+n\gamma)=0;$$

and the polar of the same point (x, y, z) with respect to (Q) will be the second chord of intersection

$$(lx+my+nz)\left[\alpha(g-d)+\beta(h-e)+\gamma(k-f)\right]=0;$$

provided that the following relations subsist between x, y, z:—

$$(2u+lg+ld)x+(2w'+lh+md)y+(2v'+lk+nd)z=0,$$

$$(2w'+mg+le)x+(2v+mh+me)y+(2u'+mk+ne)z=0,$$

$$(2v'+ng+lf)x+(2u'+nh+mf)y+(2w+nk+nf)z=0.$$

The determinant, which expresses their compatibility, is the required condition for anorthotomic intersection.

This determinant may be given as the difference of two others

$$2K\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ d+g, & e+h, & f+k, & -1 \end{vmatrix} - \begin{vmatrix} U, & W', & V', & nh-mk \\ W', & V, & U', & lk-ng \\ V', & U', & W, & mg-lh \\ ne-mf, & lf-nd, & md-le, & 2 \end{vmatrix} = 0.$$

The polars of another point with respect to (Q) and (P) will be the same chords in reverse order—under the same condition.

4. A binodal quartic may be generated in four different ways, as the envelope of a variable conic (G) always passing through two fixed points, and intersecting anorthogonally a given conchordal conic (J) through D, E, so that the pole of the chord DE with respect to G moves along a given dirigent conic.

5. The following system of conchordal conics is coanorthotomic,

and each is the Jacobian of the other three, and O, A, B, C are the poles of the common chord DE with respect to the Jacobians:—

$$ua^2\alpha^2+vb^2\beta^2+wc^2\gamma^2=0 \quad \ldots\ldots(J),$$

$$la\,(ua^2\alpha^2+vb^2\beta^2+wc^2\gamma^2)=2ua^2\alpha\,(laa+mb\beta+nc\gamma) \quad \ldots\ldots(J_1),$$

$$mb\Sigma ua^2\alpha^2=2vb^2\beta\Sigma laa \quad \ldots\ldots(J_2),$$

$$nc\Sigma ua^2\alpha^2=2wc^2\gamma\Sigma laa \quad \ldots\ldots(J_3).$$

The conditions of coanorthotomy are satisfied; for the two polars of the two points A and $0\left(\frac{l}{ua}, \frac{m}{vb}, \frac{n}{wc}\right)$ with respect to J and J_1 are the same, when taken in turns, viz., their common chords BC and DE $(laa+mb\beta+nc\gamma=0)$.

So also the two polars of A and B with respect to J_1, J_2 are their two common chords, when taken in turns.

6. The four corresponding dirigent conics are

$$La^2\alpha^2+Mb^2\beta^2+Nc^2\gamma^2=0 \quad \ldots\ldots(F),$$

or, in the tangential form,

$$\frac{p^2}{L}+\frac{q^2}{M}+\frac{r^2}{N}=0 \quad \ldots\ldots(F),$$

and three others F_1, F_2, F_3, corresponding to J_1, J_2, J_3, will be proved to be of the type

$$\frac{p^2}{L}+\frac{q^2}{M}+\frac{r^2}{N}+r\omega_1\omega_2=0.$$

That is, all four dirigent conics will be proved to be inscribed in the same quadrilateral, of which the two fixed points D, E (here denoted by ω_1, ω_2) are vertices.

These two points (ω_1, ω_2), through which all the Jacobians pass, are thus denoted by tangential coordinates

$$\left(\frac{lp}{u}+\frac{mq}{v}+\frac{nr}{w}\right)^2-\left(\frac{p^2}{u}+\frac{q^2}{v}+\frac{r^2}{w}\right)\left(\frac{l^2}{u}+\frac{m^2}{v}+\frac{n^2}{w}\right)=\omega_1\omega_2.$$

7. To determine the equation to a binodal quartic, which is generated by a variable conic (G), coanorthotomic with the conchordal (J), in such a way that the pole (λ, μ, ν) of the common chord with respect to (G) moves on a dirigent conic (F).

Such a variable conic is thus denoted:

$$(\lambda la+\mu mb+\nu nc)\,\Sigma ua^2\alpha^2=2\,(u\lambda a^2\alpha+v\mu b^2\beta+w\nu c^2\gamma)\,\Sigma laa \ldots(G),$$

or, as it may be written,

$$\lambda J_1+\mu J_2+\nu J_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (G).$$

For the two polars of the pole (λ, μ, ν) and of $0\left(\frac{l}{u}, \frac{m}{v}, \frac{n}{w}\right)$ for G and J, when taken with respect to J and G by turns, are the same, viz., their two common chords.

Since the pole (λ, μ, ν) lies on (F),

$$La^2\lambda^2+Mb^2\mu^2+Nc^2\nu^2=0.$$

The variation of G, subject to this condition, determines the binodal quartic, whose nodes are D, E,

$$\frac{J_1^2}{La^2}+\frac{J_2^2}{Mb^2}+\frac{J_3^2}{Nc^2}=0.$$

8. If the quartic be compared with the tangential form of (F), a quadric transformation is recognised,

$$J_1 : J_2 : J_3 = ap : bq : cr.$$

In the special case of bicircular quartics, l, m, n become 1, 1, 1; u, v, w become $\cot A$, $\cot B$, $\cot C$; a factor is removed in each case, and

$$J_1 : J_2 : J_3 = p : q : r.$$

This is Dr. Casey's theorem for quadric transformation.

9. The following identities connect the four Jacobians :—

$$\frac{l}{ua}J_1+\frac{m}{vb}J_2+\frac{n}{wc}J_3+2(laa+mb\beta+nc\gamma)^2=\left(\frac{l^2}{u}+\frac{m^2}{v}+\frac{n^2}{w}\right)J,$$

$$\frac{1}{ua^2}J_1^2+\frac{1}{vb^2}J_2^2+\frac{1}{wc^2}J_3^2=J^2\Sigma\frac{l^2}{u}.$$

10. The poles of the common chord DE for the four (J) conics being O, A, B, C, the four triangles ABC, OBC, OCA, OAB are autopolar with respect to the four (J) and the corresponding four (F) conics, respectively.

This theorem is established geometrically by Dr. Casey for Bicircular Quartics. See his Memoir, § 24.

11. To show that the same binodal quartic may be generated by another variable conic (G_1), coanorthotomic with (J_1), and having its pole of DE on (F_1).

It will be necessary first to establish the equation to F_1, which

was assumed in § 6 to be

$$\frac{p}{L}+\frac{q^2}{M}+\frac{r^2}{N}+r\omega_1\omega_2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (F_1).$$

Assume its equation, when referred to its autopolar triangle OBC, to be

$$\frac{P^2}{L_1}+\frac{q^2}{N_1}+\frac{r^2}{M_1}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (F_1).$$

Then P is thus related to the coordinate system of ABC :

$$P\left(\frac{l}{u}+\frac{m}{v}+\frac{n}{w}\right)=\frac{lp}{u}+\frac{mq}{v}+\frac{nr}{w};$$

or, in terms of the fixed points $\omega_1\omega_2$, by § 6,

$$P^2\left(\Sigma\frac{l}{u}\right)^2=\omega_1\omega_2+\left(\frac{p^2}{u}+\frac{q^2}{v}+\frac{r^2}{w}\right)\Sigma\frac{l^2}{u}.$$

Substitute for P in (F_1), and compare the coefficients in the two assumed forms of (F_1) :

$$r\Sigma\frac{l^2}{u}=\frac{u}{L}, \quad \frac{1}{M}=\frac{1}{L}\frac{u}{v}+\frac{1}{N_1}, \quad \frac{1}{N}=\frac{1}{L}\frac{u}{w}+\frac{1}{M_1},$$

if

$$\frac{L_1}{L}\left(\Sigma\frac{l}{u}\right)^2=\frac{1}{u}\Sigma\frac{l^2}{u}.$$

If these values of L_1, M_1, N_1, and those of $\omega_1\omega_2$, be replaced in (F_1), it becomes

$$\frac{p^2}{L}+\frac{q^2}{M}+\frac{r^2}{N}-\frac{u}{L}\Sigma\frac{p^2}{u}+r\left(\Sigma\frac{lp}{u}\right)^2=0 \quad \ldots\ldots\ldots\ldots (F_1),$$

or, if for brevity we write

$$\frac{r}{r_2}=\frac{1}{M}-\frac{u}{Lv}, \quad \frac{r}{r_3}=\frac{1}{N}-\frac{u}{Lw},$$

$$\frac{1}{r_2}q^2+\frac{1}{r_3}r^2+\left(\frac{lp}{u}+\frac{mq}{v}+\frac{nr}{w}\right)^2=0 \quad \ldots\ldots\ldots\ldots\ldots (F_1).$$

Both this form and its equivalent equation in point-coordinates exhibit OBC as an autopolar triangle,

$$a^2+r_2b^2\left(\frac{ma}{vb}-\frac{l\beta}{ua}\right)^2+r_3c^2\left(\frac{na}{wc}-\frac{l\gamma}{ua}\right)^2=0 \quad \ldots\ldots\ldots\ldots\ldots (F_1).$$

The Jacobian

$$la\,(ua^2\alpha^2+vb^2\beta^2+wc^2\gamma^2)=2ua^2\alpha\,(la\alpha+mb\beta+nc\gamma) \quad \ldots\ldots (J_1)$$

may also be written to exhibit OBC as autopolar,

$$-a^2\Sigma\frac{l^2}{u}+vb^2\left(\frac{ma}{vb}-\frac{l\beta}{ua}\right)^2+wc^2\left(\frac{na}{wc}-\frac{l\gamma}{ua}\right)^2=0 \quad \ldots\ldots\ldots(J_1).$$

The generating conic, which is coanorthotomic with J_1, is

$$\lambda J_1+\mu J_2+\nu J_3+\frac{2}{l}\lambda ua\,(laa+mb\beta+nc\gamma)^2=0 \quad \ldots\ldots\ldots(G_1).$$

For the two polars of the pole of DE for G_1, viz. (λ, μ, ν), and of A when taken with respect to J_1 and G_1 by turns, are their two common chords DE and the line

$$l\Sigma ua^2a\lambda=ua\,a\Sigma la\lambda+ua\lambda\Sigma laa.$$

Since the pole (λ, μ, ν) is to lie on (F_1),

$$\lambda^2+r_2b^2\left(\frac{m\lambda}{vb}-\frac{l\mu}{ua}\right)^2+r_3c^2\left(\frac{n\lambda}{wc}-\frac{l\nu}{ua}\right)^2=0.$$

The envelope of G_1, subject to this condition, is the binodal quartic.

By differentiating G_1 and F_1, we find

$$J_1+\frac{2ua}{l}(\Sigma laa)^2\propto\lambda+\frac{mbr_2}{v}\left(\frac{m\lambda}{vb}-\frac{l\mu}{ua}\right)+\frac{ncr_3}{w}\left(\frac{n\lambda}{wc}-\frac{l\nu}{ua}\right),$$

$$J_2\propto-\frac{lb^2r_2}{ua}\left(\frac{m\lambda}{vb}-\frac{l\mu}{ua}\right).$$

$$J_3\propto-\frac{lc^2r_3}{ua}\left(\frac{n\lambda}{wc}-\frac{l\nu}{ua}\right).$$

Multiply by $\frac{l}{ua}, \frac{m}{vb}, \frac{n}{wc}$ in succession, and add; by § 9,

$$J\Sigma\frac{l^2}{u}\propto\frac{\lambda l}{ua}.$$

Hence

$$J_2+\frac{mbr_2}{v}J\Sigma\frac{l^2}{u}\propto\frac{b^2}{u^2a^2}\mu l^2r_2,$$

$$J_3+\frac{ncr_3}{w}J\Sigma\frac{l^2}{u}\propto\frac{c^2}{u^2a^2}\nu l^2r_3.$$

Substitute these values of λ, μ, ν in (G_1), the envelope may, by the aid of § 9, be thus expressed:

$$J^2\left(\Sigma\frac{l^2}{u}\right)^2+\frac{1}{b^2r_2}J_2^2+\frac{1}{c^2r_3}J_3^2=0.$$

But, by the second identity given in § 9,

$$J^2\Sigma\frac{l^2}{u}=\frac{1}{ua^2}J_1^2+\frac{1}{vb^2}J_2^2+\frac{1}{wc^2}J_3^2.$$

Therefore, after replacing also the values of r_1, r_2, and r, we have

$$\frac{u}{L}\left(\frac{1}{ua^2}J_1^2+\frac{1}{vb^2}J_2^2+\frac{1}{wc^2}J_3^2\right)+\frac{1}{b^2}J_2^2\left(\frac{1}{M}-\frac{u}{Lv}\right)+\frac{1}{c^2}J_3^2\left(\frac{1}{N}-\frac{u}{Lw}\right);$$

that is,

$$\frac{1}{La^2}J_1^2+\frac{1}{Mb^2}J_2^2+\frac{1}{Nc^2}J_3^2=0,$$

the same form which was found in § 7. The assumptions made with reference to the forms of the (F) dirigent conics are therefore verified.

12. To determine, in binodal quartics, the property of which anallagmatism in bicircular quartics is a particular case.

Let a transversal pass through O the pole of DE for J, whose coordinates are given in § 5,

$$\frac{uaf}{l}=\frac{vbg}{m}=\frac{wch}{n}=\frac{ua^2\lambda f+vb^2\mu g+wc^2\nu h}{al\lambda+bm\mu+cn\nu}=\frac{ua^2f^2+vb^2g^2+wc^2h^2}{laf+mbg+nch}.$$

If λ, μ, ν are the direction-sines of a transversal,

$$\frac{\alpha-f}{\lambda}=\frac{\beta-g}{\mu}=\frac{\gamma-h}{\nu}=\rho.$$

To determine its intersections with J_1,

$$ua^2(\lambda\rho+f)^2+vb^2(\mu\rho+g)^2+wc^2(\nu\rho+h)^2$$
$$=\frac{2ua}{l}(\lambda\rho+f)(\rho\Sigma la\lambda+\Sigma lfa)\ldots\ldots\ldots\ldots\ldots(\mathrm{A}).$$

Next, to determine its intersections with J,

$$ua^2(\lambda R+f)^2+vb^2(\mu R+g)^2+wc^2(\nu R+h)^2=0.$$

Substitute in (A) the two values R_1, R_2 thence derived, and remember the various values of $\frac{uaf}{l}$,

$$\rho^2-(R_1+R_2)\rho+R_1R_2=\left(\frac{\lambda\rho}{f}+1\right)\left[-\rho(R_1+R_2)+2R_1R_2\right],$$

or $$\rho^2 - R_1 R_2 = \frac{\lambda\rho}{f}\left[-\rho\,(R_1+R_2)+2R_1R_2\right].$$

Similar values of the vector are found for the intersections with J_2, J_3, by writing $\frac{\mu}{g}$, $\frac{\nu}{h}$, instead of $\frac{\lambda}{f}$.

We have to determine such a relation between these values of ρ and those of another σ on the same transversal, that the one may be substituted for the other, without altering the binodal quartic

$$\frac{1}{la^2}J_1^2+\frac{1}{mb^2}J_2^2+\frac{1}{nc^2}J_3^2=0;$$

that is, $$\rho^2-R_1R_2 : \sigma^2-R_1R_2 = \rho\left[-\rho\,(R_1+R_2)+2R_1R_2\right] : \sigma\left[-\sigma\,(R_1+R_2)+2R_1R_2\right].$$

This relation may take the simpler form

$$(2\rho-R_1-R_2)(2\sigma-R_1-R_2) = (R_1-R_2)^2.$$

The values of R_1, R_2 vary with the direction of the transversal, whereas in the case of the bicircular quartic, where $R_1+R_2=0$, the relation of inversion ($\rho\sigma = R^2$) is immutable.

There are four centres or poles of this modified form of inversion, viz., O, A, B, C, the poles of DE the common chord of contact for the several Jacobians.

13. To determine the equivalent tangential equation of the binodal quartic of § 7,

$$\frac{1}{La^2}J_1^2+\frac{1}{Mb^2}J_2^2+\frac{1}{Nc^2}J_3^2=0.$$

The tangential equation of a quartic (U) is a function of (S), (T), its quartic and sextic contravariants,

$$S^3-27T^2=0.$$

It has been proved (*Quarterly Journal of Mathematics*, Vol. XXIV., p. 251) that the nodes of S and T are identical with those of U, but that in the case of T only these are biflecnodes. In binodal quartics these nodes are the fixed points D, E, (ω_1, ω_2); and all four (F) dirigent conics have been shown in § 11 to be inscribed in the same quadrilateral, of which D, E are vertices; *i.e.*, their equations differ by a multiple of $\omega_1\omega_2$.

Hence it was anticipated that S and T, when taken of U (Salmon's

Higher Plane Curves, p. 251), could be expressed so as to exhibit this nodal property.

The tangential equation required may be deduced from that to the bicircular given in the above-cited Vol. XXIV., p. 253, by making the indicated substitutions. Denote by

$$\frac{\lambda^2\mu^2\nu^2}{uvw}\,2\Sigma = \left(\frac{\lambda^2}{u}+\frac{\mu^2}{v}+\frac{\nu^2}{w}\right)\left(\frac{p^2}{u}+\frac{q^2}{v}+\frac{r^2}{w}\right)-\left(\frac{\lambda p}{u}+\frac{\mu q}{v}+\frac{\nu r}{w}\right)^2$$

$$= \omega_1\omega_2, \text{ by } \S\,6,$$

$$\frac{\lambda^2\mu^2\nu^2}{uvw}\,\theta = \left(\frac{\lambda^2}{u}+\frac{\mu^2}{v}+\frac{\nu^2}{w}\right)\left(\frac{p^2}{L}+\frac{q^2}{M}+\frac{r^2}{N}\right)-\left(\frac{u}{L}+\frac{v}{M}+\frac{w}{N}\right)\omega_1\omega_2,$$

$$LMN\,\frac{\lambda^2\mu^2\nu^2}{uvw}\cdot\frac{\phi}{2\Sigma} = \left(\frac{\lambda^2}{u}+\frac{\mu^2}{v}+\frac{\nu^2}{w}\right)\left[L(wq^2+vr^2)+M(ur^2+wp^2)\right.$$
$$\left.+N(vp^2+uq^2)\right]-(Lvw+Mwu+Nuv)\,\omega_1\omega_2,$$

$$LMN\,\frac{\lambda^2\mu^2\nu^2}{u^2v^2w^2}\cdot\frac{\chi}{4\Sigma^2} = 2\left(\frac{\lambda^2}{L}+\frac{\mu^2}{M}+\frac{\nu^2}{N}\right)\left(\frac{Lp^2}{u^2}+\frac{Mq^2}{v^2}+\frac{Nr^2}{w^2}\right)$$
$$-\left(\frac{\lambda p}{u}+\frac{\mu q}{v}+\frac{\nu r}{w}\right)^2.$$

With these values of θ, ϕ, χ, and 2Σ,

$$\tfrac{1}{3}S = \theta^2+3\phi\,;$$

that is,
$$\tfrac{1}{3}S = \left(\frac{p^2}{L}+\frac{q^2}{M}+\frac{r^2}{N}\right)^2\left(\frac{vw}{\mu^2\nu^2}+\frac{wu}{\nu^2\lambda^2}+\frac{uv}{\lambda^2\mu^2}\right)$$
$$-\left[\frac{p^2}{L}\left(\frac{2u}{L}-\frac{v}{M}-\frac{w}{N}\right)+\frac{q^2}{M}\left(\frac{2v}{M}-\frac{w}{N}-\frac{u}{L}\right)+\frac{r^2}{N}\left(\frac{2w}{N}-\frac{u}{L}-\frac{v}{M}\right)\right]2\Sigma$$
$$+\left(\frac{u^2}{L^2}+\frac{v^2}{M^2}+\frac{w^2}{N^2}-\frac{vw}{MN}+\frac{wu}{NL}-\frac{uv}{LM}\right)\left(\frac{vw}{\mu^2\nu^2}+\frac{wu}{\nu^2\lambda^2}+\frac{uv}{\lambda^2\mu^2}\right)^{-1}4\Sigma^2.$$

$$T = \theta^3+\tfrac{9}{2}\theta\phi+\tfrac{27}{8}\chi.$$

The equivalent tangential form for the binodal U, S^3-27T^2, may be simplified, after removing the factor $4\Sigma^2$ or $(\omega_1\omega_2)^2$, to

$$\theta^2(\tfrac{1}{4}\phi^2-\theta\chi)+\phi^3-\tfrac{9}{2}\theta\phi\chi-\tfrac{27}{16}\chi^2 = 0.$$

The form of the conic θ shows that it may be inscribed in the same quadrilateral as the four dirigent conics.

The nodes D, E are common to S, T, U; the nodal tangents in all

three curves constitute the quadrilateral, which touches the four F conics; but no single tangent can be drawn from D, E to touch S, T.

In U, besides the same nodal tangents, four single tangents can be drawn from D, E each to U, *i.e.*, to the class-quartic

$$\phi^2 - 4\theta\chi = 0.$$

These two pencils of single tangents are homographic. (Prof. Cayley, *Higher Plane Curves*, p. 232.)

14. Bicuspidal quartics are generated from conchordal conics, by coanorthotomy, when the pole of their common chord with respect to the generating variable conic lies on another conchordal conic.

Assume, as in § 7, the fixed conic and generator to be

$$r\alpha^2 + s\beta^2 + t\gamma^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (J),$$

$$J\Sigma l\lambda - 2\Sigma\lambda r\alpha\,\Sigma la = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (G),$$

or

$$\lambda J_1 + \mu J_2 + \nu J_3 = 0.$$

They satisfy the conditions of coanorthotomy (§ 7).

The pole of the common chord (λ, μ, ν) must lie on a dirigent conchordal conic

$$J + (d\alpha + e\beta + f\gamma)(l\alpha + m\beta + n\gamma) = 0 \quad \ldots\ldots\ldots\ldots\ldots (F).$$

Write this for the moment

$$(u, v, w, u', v', w' \mathbin{\text{⨅}} \alpha, \beta, \gamma)^2 = 0,$$

and its tangential equivalent

$$(U, \ldots\ldots W' \mathbin{\text{⨅}} ap, bq, cr)^2 = 0.$$

The quartic thus generated is, by § 8,

$$(U, \ldots\ldots W' \mathbin{\text{⨅}} J_1, J_2, J_3)^2 = 0,$$

or $(U, \ldots\ldots W' \mathbin{\text{⨅}} lJ - 2r\alpha\Sigma la,\ mJ - 2s\beta\Sigma la,\ nJ - 2s\gamma\Sigma la)^2 = 0.$

This may be expanded, and take the form

$$\left\{J\Sigma(Ul^2 + 2U'mn) - 2\Sigma la\,\Sigma\left[r\alpha\,(lU + mW' + nV')\right]\right\}^2$$
$$+ 4K(\Sigma la)^2\,\Sigma\left[r^2\alpha^2(m^2w + n^2v - 2mnu')\right.$$
$$\left. - 2st\beta\gamma\,(umn - v'lm - w'ln + u'l^2)\right] = 0,$$

where K is the discriminant of F.

When the values of $u, v, \ldots$ are substituted, the coefficient of K becomes

$$-rst\,(\Sigma la)^2+J\Sigma\,(l^2st).$$

The quartic is seen to be bicuspidal when it is written under the form

$$\{DJ-2\Sigma la\,\Sigma\theta a\}^2$$
$$+\frac{4K}{D}\,(\Sigma la)^2\{-Drst\,(\Sigma la)^2+2\Sigma la\,\Sigma\theta a\,\Sigma l^2st+(DJ-2\Sigma la\,\Sigma\theta a)\,\Sigma l^2st\}=0.$$

15. Bicuspidal quartics, whose cusps are at infinity, may be generated by coanorthotomy from a system of homothetic conics.

Let there be three homothetic conics,

$$k^2\,(x+a)^2+l^2y^2=s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (J),$$
$$k^2x^2+l^2y^2=r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F),$$
$$k^2\,(x-a)^2+l^2\,(y-\beta)^2=\rho^2 \ldots\ldots\ldots\ldots\ldots\ldots (G).$$

The variable conic (G) is to be coanorthotomic with (J), and its centre constrained to lie on (F), (J) and (F) being fixed; the envelope of G is a bicuspidal quartic. By § 2, Def., the centre of (J) is the pole for G of their common chord, if the condition of coanorthotomy be

$$k^2\,(a+a)^2+l^2\beta^2=\rho^2+s^2$$
$$=k^2\,(x-a)^2+l^2\,(y-\beta)^2+s^2.$$

Hence $$2ak^2\,(x+a)+2\beta l^2y=k^2x^2+l^2y^2-k^2a^2+s^2 \ldots\ldots\ldots\ldots (1).$$

Also, $$k^2a^2+l^2\beta^2=r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The envelope of (1), subject to the condition (2), is

$$k^2x^2+l^2y^2-k^2a^2+s^2=2r\sqrt{\left[k^2\,(x+a)^2+l^2y^2\right]},$$

or $$(k^2x^2+l^2y^2-k^2a^2+s^2-2r^2)^2=4r^2\,(r^2-s^2)\,8k^2a^2r^2+8k^2ar^2x.$$

In Fig. 1, $k^2=1$, $l^2=-1$; so that the cuspidal tangents at D, E intersect in C, the origin; AC, BC are harmonic conjugates of the cuspidal tangents.

The equation to the bicuspidal quartic may be written

$$(x^2-y^2-p_2)^2=4p_3\,(2x+p_1).$$

The constants are selected combinations of d, e, f, that is, of OF_1, OF_2, OF_3:

$$p_1=d+e+f,\quad p_2=ef+fd+de,\quad p_3=def.$$

From each cusp three (parallel) single tangents can be drawn,

$$DF_1,\ DF_2,\ DF_3;\quad EF_1,\ EF_2,\ EF_3.$$

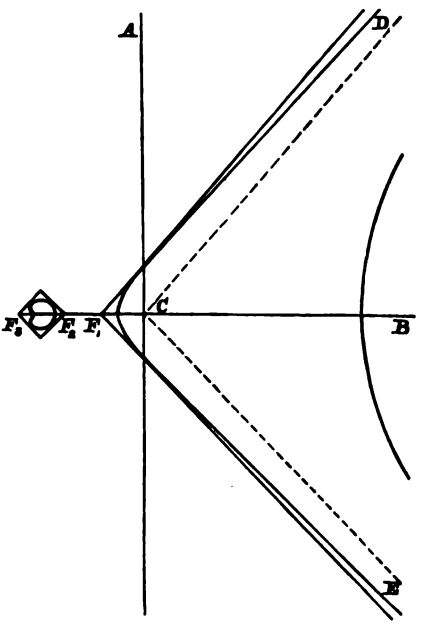

FIG. 1.

Their equations are

$$x \pm y + d = 0,\quad x \pm y + e = 0,\quad x \pm y + f = 0.$$

For if the line $(x+y+\theta = 0)$ be drawn from a cusp, it will meet the quartic in two coincident points, if

$$\theta^3 - p_1\theta^2 + p_2\theta - p_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

The discriminant of (A) is also that of the quartic

$$(x^3 - p_3)^2 = 4p_3\,(2x + p_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B).$$

Since the reducing cubic, in Euler's solution, is

$$z^3 - p_2 z^2 + p_1 p_3 z - p_3^2 = 0,$$

the roots of (B) are $\pm\sqrt{(ef)} \pm \sqrt{(fd)} \pm \sqrt{(de)}$.

There are therefore four apses of the figure, thus defined, besides the bitangent $(2x+p_1)$.

The assumed quartic is the expansion, when cleared of radicals, of the following form :—

$$\sqrt{d}\,(e-f)\,\sqrt{\left[k^2(x+d)^2-l^2y^2\right]}+\sqrt{e}\,(f-d)\,\sqrt{\left[k^2(x+e)^2-l^2y^2\right]}$$
$$+\sqrt{f}\,(d-e)\,\sqrt{\left[k^2(x+f)^2-l^2y^2\right]}=0.$$

This form exhibits the property

$$L\sqrt{(p_1q_1)}+M\sqrt{(p_2q_2)}+N\sqrt{(p_3q_3)}=0,$$

where p, q denote pairs of perpendiculars drawn from a current point of the quartic on the single tangents drawn from the cusps; L, M, N are known functions of d, e, f.

16. In the general equation of § 15, if $k^2=1$, and $l^2=1$, the cusps are imaginary, and the Cartesian is given, whose four collinear cusps correspond to the intersections of pairs of tangents in § 15, when $l^2=-1$.

Just as the Cartesian becomes the Limaçon and Cardioid, when two and three of the focal distances from the triple focus coincide, so this bicuspidal quartic becomes nodal and tricuspidal, according as two and three distances d, e, f become equal.

17. The two nodes D, E, or the points in which the chord $(l\alpha+m\beta+n\gamma)$ intersects the conic S (of § 3) will be real, coincident, or imaginary, according as $(U, \ldots W\,\mathfrak{J}\,l, m, n)^2 <=> 0$.

In the second case, the generated quartic is tacnodal.

II. *On Binodal Quartics, which are to be drawn so as to have quadruple contact with a given conic in four assigned points.*

18. Def.—Two conics, which have double contact with another conic, intersect each other anorthogonally, and such intersection is anorthotomic, when the centre of contact for one conic is the pole with respect to the other conic of a chord of intersection of the two conics.

19. To find the condition of anorthotomy for the two conics

$$l\alpha+m\beta+n\gamma=S^1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$\lambda\alpha+\mu\beta+\nu\gamma=S^1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where S is the conic

$$(u, v, w, u', v', w'\,\mathfrak{J}\,\alpha, \beta, \gamma)^2=0.$$

The centre of contact in (1), or pole of $(l\alpha+m\beta+n\gamma=0)$ with respect to S, is

$$\frac{\alpha'}{Ul+W'm+V'n}=\frac{\beta'}{W'l+Vm+U'n}=\frac{\gamma'}{V'l+U'm+Wn},$$

where $U, \ldots, U', \ldots$ are the minors $vw-u'^2, \ldots, v'w'-uu', \ldots$. The polar of this point with respect to (2) is

$$\alpha\left[(\lambda^2-u)\alpha'+(\lambda\mu-w')\beta+(\lambda\nu-v')\gamma\right]+\ldots=0.$$

This may be written, if K denote the discriminant of S,

$$\{\alpha(\lambda-l)+\beta(\mu-m)+\gamma(\nu-n)\}K$$
$$+(\lambda\alpha+\mu\beta+\nu\gamma)\{-K+Ul\lambda+\ldots+U'(m\nu+n\mu)+\ldots\}=0.$$

The factor of $(\lambda\alpha+\mu\beta+\nu\gamma)$ may take the determinant form

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ \lambda, & \mu, & \nu, & 1 \end{vmatrix}$$

If this determinant be zero, it is the required condition of coanorthotomy; for a chord of intersection is the polar required. From its symmetry, the same condition is necessary and sufficient that the centre of contact of (2) is the pole with respect to (1) of that same chord of intersection.

Dr. Casey gave this condition (*Bicircular Quartics*, § 129), and thence deduced the theorem on which the definition (§ 18) depends.

Cor.—The conditions of coanorthotomy for the two conics,

$$l\alpha+m\beta+n\gamma=S^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\lambda S_1+\mu S_2+\nu S_3=S^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

is

$$l\lambda+m\mu+n\nu=1,$$

if

$$2S_1=\frac{dS}{d\alpha},\quad 2S_2=\frac{dS}{d\beta},\quad 2S_3=\frac{dS}{d\gamma}.$$

Hence a conic (2) can readily be found, which shall be coanorthotomic with three others of the form (1).

20. The following forms* of coanorthotomic Jacobians, which are employed to generate binodal quartics touching a given conic in four points, were suggested by the corresponding coorthotomic Jacobians, which are used to generate sphero-quadrics (*Proceedings of the London Mathematical Society*, Vol. XVI., p. 116):—

$$S_1-\frac{1}{u'}S^{\frac{1}{2}}\sqrt{(u'v'w')}=0\ldots\ldots\ldots\ldots\ldots\ldots(J_1),$$

$$S_2-\frac{1}{v'}S^{\frac{1}{2}}\sqrt{(u'v'w')}=0\ldots\ldots\ldots\ldots\ldots\ldots(J_2),$$

* See below, § 33, for a second set of coanorthotomic Jacobians, which are applicable to the genesis of binodal quartics.

$$S_3 - \frac{1}{w'} S^2 \sqrt{(u'v'w')} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (J_4),$$

$$v'w'\alpha + w'u'\beta + u'v'\gamma - S^2\sqrt{(u'v'w')} = 0 \ldots\ldots\ldots\ldots (J).$$

This last Jacobian (J) may be written in the forms

$$\frac{U'}{u'}\alpha^2 + \frac{V'}{v'}\beta^2 + \frac{W'}{w'}\gamma^2 = 0; \quad \alpha J_1 + \beta J_2 + \gamma J_3 = 0.$$

It will be seen that the vertices of ABC are the centres of contact of J_1, J_2, J_3; and also centres of symptosis of the chords of intersection of J_2, J_3, J; J_3, J_1, J; J_1, J_2, J. respectively.

The point $(U'\alpha = V'\beta = W'\gamma)$ is O, the centre of contact of J and also a centre of symptosis of the chords of intersection of J_1, J_2, J_3.

21. The centres of contact A, B, C, O of the four conics J_1, J_2, J_3, J_4 are the vertices of the quadrangle, in which any J and F (§ 24) pair intersect, and, taken by triads, are the vertices of triangles, which are autopolar with respect to both the J and F conics.

In the accompanying figure (2), S is a circumscribed conic, and the centres of contact A, B, C become points of contact.

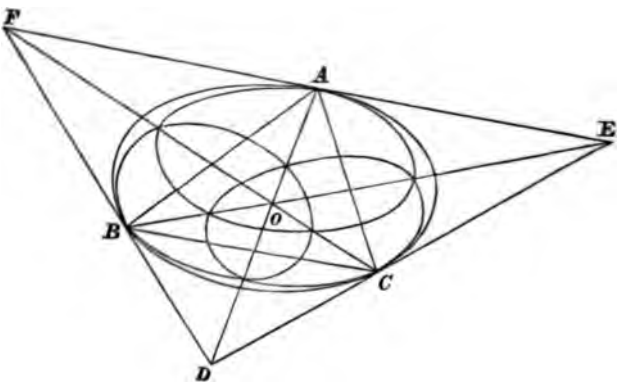

Fig. 2.

The Jacobian (J) is in this case imaginary, although its chord of contact with S is real, viz., the harmonic of 0, its centre of contact, with respect to ABC.

22. To find the equation to a binodal quartic, which touches the conic (S), by considering it as the envelope of a variable conic, which has double contact with (S), and whose centre of contact moves on a dirigent conic (F), and which cuts another conic (J), also having double contact with (S), anorthogonally.

The equation to the generating conic, whose centre of contact is (λ, μ, ν), is

$$\lambda S_1 + \mu S_2 + \nu S_3 = S^1,$$

where S_1, S_2, S_3, as before, are the partial differential coefficients of S.

The dirigent conic (F), when referred to the same autopolar triangle as (J), is

$$la^2\alpha^2 + mb^2\beta^2 + nc^2\gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (F).$$

Since the centre of contact is constrained to lie on it,

$$la^2\lambda^2 + mb^2\mu^2 + nc^2\nu^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The condition of coanorthotomy of the generator with J is, by § 19 *Cor.*,

$$\frac{\lambda}{u'} + \frac{\mu}{v'} + \frac{\nu}{w'} = \frac{1}{\sqrt{(u'v'w')}}.$$

The equation to the generator may assume a linear form in λ, μ, ν,

$$\lambda S_1 + \mu S_2 + \nu S_3 - \sqrt{(u'v'w')}\left(\frac{\lambda}{u'} + \frac{\mu}{v'} + \frac{\nu}{w'}\right) S^1 = 0,$$

or

$$\lambda J_1 + \mu J_2 + \nu J_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The envelope of (2), subjected to the condition (1), gives the binodal quartic

$$\frac{1}{la^2} J_1^2 + \frac{1}{mb^2} J_2^2 + \frac{1}{nc^2} J_3^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let this be compared with the tangential equation to the dirigent (F)

$$\frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} = 0;$$

it appears that the quadric transformation from the dirigent conic to the binodal quartic is here also

$$ap : bq : cr :: J_1 : J_2 : J_3.$$

23. If the values of J_1, J_2, J_3 be substituted in (3), and the result expanded, it will be seen that the conic of contact (S) touches the quartic in four points, in which it intersects the conic

$$\frac{1}{la^2} S_1^2 + \frac{1}{mb^2} S_2^2 + \frac{1}{nc^2} S_3^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and that the line of nodes is

$$\frac{1}{la^2 u'} S_1 + \frac{1}{mb^2 v'} S_2 + \frac{1}{nc^2 w'} S_3 = 0.$$

The four points of contact of the binodal with S are those at which S and F have common tangents.

This is seen by comparing the coefficients of their common tangents

$$xS_1+yS_2+zS_3=0; \quad la^2x\alpha_1+mb^2\beta y_1+nc^2z\gamma_1=0,$$

$$\frac{1}{S_1}la^2\alpha_1=\frac{1}{S_2}mb^2\beta_1=\frac{1}{S_3}nc^2\gamma_1.$$

Substitute these values of $\alpha_1, \beta_1, \gamma_1$ in F:

$$\frac{1}{la^2}S_1^2+\frac{1}{mb^2}S_2^2+\frac{1}{nc^2}S_3^2=0.$$

The intersection of this conic with S determines the points of contact.

24. The following identity connects the Jacobians of §20:—

$$\frac{u'}{U'}J_1^2+\frac{v'}{V'}J_2^2+\frac{w'}{W'}J_3^2=\frac{K}{U'V'W'}J^2.$$

This is proved by comparing the coefficients of the variables on either side by aid of the identities

$$u'\left(\frac{u}{U'}+\frac{w'}{V'}+\frac{v'}{W'}\right)=v'\left(\frac{w'}{U'}+\frac{v}{V'}+\frac{u'}{W'}\right)=w'\left(\frac{v'}{U'}+\frac{u'}{V'}+\frac{w'}{W'}\right)$$

$$=\frac{uu'}{U'}+\frac{vv'}{V'}+\frac{ww'}{W'}+2=\frac{u'v'w'}{U'V'W'}K,$$

and

$$\frac{1}{u'U'}+\frac{1}{v'V'}+\frac{1}{w'W'}=\frac{1}{u'v'w'}+\frac{K}{U'V'W'}.$$

Hence

$$\frac{S_1}{U'}+\frac{S_2}{V'}+\frac{S_3}{W'}=\frac{K}{U'V'W'}(\alpha v'w'+\beta w'u'+\gamma u'v')$$

$$\frac{u'}{U'}S_1^2+\frac{v'}{V'}S_2+\frac{w'}{W'}S_3^2=\frac{K}{U'V'W'}(\alpha v'w'+\beta w'u'+\gamma u'v')^2-S.$$

By the aid of these relations the proposed identity may be proved.

25. To express the three dirigent conics F_1, F_2, F_3 in terms of F. Assume F_1 to have the form

$$\frac{P^2}{l_1}+\frac{q^2}{n_1}+\frac{r^2}{m_1}=0,$$

when it is referred to the autopolar triangle OPC. P_1P, the perpendicular from O on any tangent of F_1, is known in terms of p, q, r:

$$P\left(\frac{a}{U'}+\frac{b}{V'}+\frac{c}{W'}\right)=\frac{ap}{U'}+\frac{bq}{V'}+\frac{cr}{W'}.$$

We have also the identity

$$U'V'W'\left(\frac{ap}{U'}+\frac{bq}{V'}+\frac{cr}{W'}\right)^2=K\left(\frac{u'}{U'}a^2p^2+\frac{v'}{V'}b^2q^2+\frac{w'}{W'}c^2r^2\right)$$
$$+\Sigma\,(Ua^2p^2+2U'bcqr).$$

The last term denotes the tangential equivalent of S, and will be written Σ. By aid of this identity the value of P may be written

$$P^2\mu=K\left(\frac{u'}{U'}a^2p^2+\frac{v'}{V'}b^2q^2+\frac{w'}{W'}c^2r^2\right)+\Sigma,$$

if
$$\mu=U'V'W'\left(\frac{a}{U'}+\frac{b}{V'}+\frac{c}{W'}\right)^2.$$

Now make this substitution in the assumed value of F_1,

$$\frac{q^2}{n_1}+\frac{r^2}{m_1}+\frac{K}{l_1\mu}\left(\frac{u'}{U'}a^2p^2+\frac{v'}{V'}b^2q^2+\frac{w'}{W'}c^2r^2\right)+\frac{1}{l_1\mu}\Sigma.$$

Assume, what will be proved below, that F_1 and F have common tangents with S or Σ; their tangential equations can therefore only differ by a multiple of Σ.

Equate the coefficients of F_1 and F, or

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0.$$

$$\frac{1}{l}=\frac{u'}{U'}\frac{a^2}{l_1\mu}K;\quad \frac{1}{m}=\frac{v'}{V'}\frac{b^2}{l_1\mu}K+\frac{1}{n_1};\quad \frac{1}{n}=\frac{w'}{W'}\frac{c^2}{l_1\mu}K+\frac{1}{m_1}.$$

The equation of F_1 may be now written

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}+\frac{U'}{u'a^2}\frac{1}{Kl}(Ua^2p^2+\ldots+2U'bcqr+\ldots)=0.$$

It may be presented conveniently in another form, if

$$K\theta=\frac{U'}{u'},\quad la^2=mb^2s=nc^2t,$$

$$a^2p^2(1+\theta U)+b^2q^2(s+\theta V)+c^2r^2(t+\theta W)$$
$$+2\theta U'bcqr+2\theta V'carp+2\theta W'abpq=0.\ldots\ldots\ldots(F_1).$$

26. To prove that the dirigent conics F, F_1, F_2, F_3 are all contangential with S.

The proof is of the converse proposition: if we assume the tangential value of F_1 given in § 25 to be true, it can be shown that the preceding binodal quartic can be generated from the Jacobian (J_1) and the dirigent (F_1) as it was generated in § 22, from J and F.

A generator of the same form as in § 22

$$\lambda S_1+\mu S_2+\nu S_3 = S^{\frac{1}{2}}$$

will cut anorthogonally

$$ua+w'\beta+v'\gamma = \frac{1}{u'}\sqrt{(u'v'w')}\,S^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (J_1),$$

if, by § 19, *Cor.*,

$$\lambda u+\mu w'+\nu v' = \frac{1}{u'}\sqrt{(u'v'w')} = \tau \text{ (suppose)};$$

so that $S^{\frac{1}{2}}(\lambda u+\mu w'+\nu v') = \tau(\lambda S_1+\mu S_2+\nu S_3) \ldots\ldots\ldots\ldots\ldots\ldots (1).$

Assume the equivalent equation to (F_1) in point-coordinates to be

$$(x, y, z, 0, y', z' \mathbin{\text{\textbigcircle}} a, \beta, \gamma)^3 = 0.$$

The coefficient of $2\beta\gamma$ is

$$\theta^2 V'W'-\theta U'(1+\theta U) = \theta^2 K\omega'-\theta U' = 0.$$

Since the centre of contact of the generator must lie on F_1,

$$(x, y, z, 0, y', z' \mathbin{\text{\textbigcircle}} \lambda, \mu, \nu)^3 = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

The envelope of (1) varying and subject to the condition (2) will be found to be the same binodal quartic.

By proceeding in the usual way,

$$\begin{aligned} x\lambda + z'\mu + y'\nu &\propto uS^{\frac{1}{2}}-\tau S_1 \\ z'\lambda + y\mu + x'\nu &\propto w'S^{\frac{1}{2}}-\tau S_2, \\ y'\lambda + x'\mu + z\nu &\propto v'S^{\frac{1}{2}}-\tau S_3. \end{aligned}$$

Then, remembering that $yz-x'^2 = 1+\theta U$, ... = ...,

$$\begin{aligned} \lambda &\propto (1+\theta U)(uS^{\frac{1}{2}}-\tau S_1)+\theta W'(w'S^{\frac{1}{2}}-\tau S_2)+\theta V'(v'S^{\frac{1}{2}}-\tau S_3) \\ &\propto \theta K(1-a\tau)+uS^{\frac{1}{2}}-\tau S_1, \\ \mu &\propto -\theta K\beta\tau+s(w'S^{\frac{1}{2}}-\tau S_2), \\ \nu &\propto -\theta K\gamma\tau+t(v'S^{\frac{1}{2}}-\tau S_3). \end{aligned}$$

Substitute these values of λ, μ, ν in (1):

$$(uS^{\frac{1}{2}}-\tau S_1)^2+s(w'S^{\frac{1}{2}}-\tau S_2)^2+t(v'S^{\frac{1}{2}}-\tau S_3)^2+\theta K(uS^{\frac{1}{2}}-2\tau S_1+\tau^2 S) = 0.$$

This may take the form

$$\left(\frac{v'w'}{u'}S^1 - rS_1\right)^2 + s\left(w'S^1 - rS_2\right)^2 + t\left(v'S^1 - rS_3\right)^2 = 0,$$

since $\theta K = \frac{U'}{u'}$, $r^2u' = v'w'$, and $\frac{U'}{u'}\left(u + \frac{v'w'}{u'}\right) + u^2 = \frac{v'^2w'^2}{u'^2}$.

Finally, it takes the preceding form of the binodal (§ 22),

$$\frac{1}{la^2}J_1^2 + \frac{1}{mb^2}J_2^2 + \frac{1}{nc^2}J_3^2 = 0.$$

The assumption is therefore true, that F_1 and F are contangential with S or Σ.

27. If a quadrilateral be circumscribed about a Jacobian and its corresponding dirigent conic, the other three dirigent conics pass through the three pairs of opposite intersections of the quadrilateral.

The tangential equations of such a J and F dirigent conic are

$$\frac{u'}{U'}a^2p^2 + \frac{v'}{V'}b^2q^2 + \frac{w'}{W'}c^2r^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (J),$$

$$\frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (F).$$

The sides of ABC are the diagonals of the circumscribed quadrilateral. The sides of the quadrilateral are denoted by tangential coordinates:

$$\frac{p^2}{l} : \frac{q^2}{m} : \frac{r^2}{n} :: \frac{1}{V'}b^2mv' - \frac{1}{W'}c^2nw'$$

$$: \frac{1}{W'}c^2nw' - \frac{1}{U'}a^2lu'$$

$$: \frac{1}{U'}a^2lu' - \frac{1}{V'}b^2mv'.$$

The sides of the quadrilateral are thus defined by point-coordinates:

$$aap \pm b\beta q \pm c\gamma r = 0.$$

Then, for two points of intersection,

$$a = 0, \quad b^2\beta^2q^2 - c^2\gamma^2r^2 = 0.$$

It will next be shown that the dirigent (F_1) passes through these

two points. By § 26 its equation is

$$(x, y, z, 0, y', z' \mathfrak{J} a, \beta, \gamma)^2 = 0.$$

At its points of intersection with BC,

$$a = 0, \quad \beta^2 y + \gamma^2 z = 0.$$

That is, if the values of y, z be derived from the coefficients of the tangential equivalent of (F_1), given in § 25,

$$\beta^2 \left[(1+\theta U)(t+\theta W) - \theta^2 V'^2\right] + \gamma^2 \left[(1+\theta U)(s+\theta V) - \theta^2 W'^2\right] = 0;$$

or, since $\theta u' K = U'$,

$$V'\beta^2 (tW' - Kw'\theta) + W'\gamma^2 (tV' - Kv'\theta) = 0.$$

That is,

$$\frac{V'\beta^2}{v'}\left(\frac{U'}{la^2u'} - \frac{W'}{nc^2w'}\right) = \frac{W'\gamma^2}{w'}\left(\frac{V'}{mb^2v'} - \frac{U'}{la^2u'}\right),$$

the same pair of intersections as before,

$$b^2\beta^2 q^2 - c^2\gamma^2 r^2 = 0.$$

Similarly, it may be shown that the dirigent conics F_2, F_3 pass through two other pairs of intersections of the circumscribed quadrilateral, which lie on AC, AB, severally.

28. To generate a binodal quartic, which shall have double contact of the third order with a given conic S.

In this case, to which Sphero-Cartesians, and all other sphero-quadrics, whose dirigents are small circles, are analogous in Spherics, the dirigent conic F must also have double contact with S.

As in § 19, let

$$S \equiv (u, v, w, u', v', w' \mathfrak{J} a, \beta, \gamma)^2 = 0,$$

$$J \equiv fa + g\beta + h\gamma - S^{\frac{1}{2}} = 0,$$

$$F \equiv ra + s\beta + t\gamma - S^{\frac{1}{2}} = 0;$$

the generating conic

$$G \equiv xS_1 + yS_2 + zS_3 - S^{\frac{1}{2}} = 0,$$

if

$$2S_1 = \frac{dS}{da}, \quad 2S_2 = \frac{dS}{d\beta}, \quad 2S_3 = \frac{dS}{d\gamma}.$$

The centre of contact of G with S is (x, y, z).

The condition of coanorthotomy of G and J is, by § 19, *Cor.*,

$$fx+gy+hz=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We have, then, to determine the envelope of the generator

$$x(S_1-fS^{\frac{1}{2}})+y(S_2-gS^{\frac{1}{2}})+z(S_3-hS^{\frac{1}{2}})=0 \quad \ldots\ldots\ldots\ldots\ldots (2),$$

subject to the condition

$$(rx+sy+tz)^2-(ux^2+\ldots+2w'xy)=0.$$

By differentiating, and using an indeterminate multiplier θ,

$$x(r^2-u)+y(rs-w')+z(rt-v')=\theta(S_1-fS^{\frac{1}{2}}),$$

$$x(rs-w')+y(s^2-v)+z(st-u')=\theta(S_2-gS^{\frac{1}{2}}),$$

$$x(rt-v')+y(st-u')+z(t^2-w)=\theta(S_3-hS^{\frac{1}{2}}).$$

The values of x, y, z hence determined, and substituted in (2), will determine the binodal quartic in terms of assigned functions of α, β, γ, viz., $S_1-fS^{\frac{1}{2}}$, ...

$$\frac{x}{\theta}=\left[U-vt^2-ws^2+2u'st\right](S_1-fS^{\frac{1}{2}})$$
$$+\left[W-u'rt-v'st+w't^2+wrs\right](S_2-gS^{\frac{1}{2}})$$
$$+\left[V-u'rs-w'st+v's^2+vrt\right](S_3-hS^{\frac{1}{2}})$$
$$=K\alpha+s^2(W'\beta-V\alpha)+t^2(V'\gamma-W\alpha)+st(-2\alpha U'+V\beta+W'\gamma)$$
$$+rt(U\gamma-V'\alpha)+rs(U\beta-W'\alpha)-c_1S^{\frac{1}{2}},$$

y, z have similar linear values.

$$\frac{1}{\theta}(ux+w'y+v'z)=Kr(r\alpha+s\beta+t\gamma)+K(u\alpha+w'\beta+v'\gamma)$$
$$-(u\alpha+w'\beta+v'\gamma)(Ur^2+\ldots+2W'rs)-(uc_1+w'c_2+v'c_3)S^{\frac{1}{2}}.$$

Hence

$$K(r\alpha+s\beta+t\gamma)^2+KS-S(Ur^2+\ldots+2W'rs)-(\lambda\alpha+\mu\beta+\nu\gamma)S^{\frac{1}{2}}$$
$$=\frac{1}{\theta}\left[x(S_1-fS^{\frac{1}{2}})+y(S_2-gS^{\frac{1}{2}})+z(S_3-hS^{\frac{1}{2}})\right]=0 \text{ by (2).}$$

The binodal quartic touches both S and F at the ends of their chord of contact, and has, therefore, double contact of the third order with S.

29. Let two lines of reference be adopted, so that, for the conic of contact,

$$S\equiv Lx^2+My^2+N.$$

Let one (J) conic be

$$fx+gy+h=S^{\frac{1}{2}},$$

whose centre of contact is

$$\left(\frac{fN}{hL}, \frac{gN}{hM}\right).$$

Let the generating conic be

$$\alpha Lx+\beta My+N=S^{\frac{1}{2}},$$

whose centre of contact is (α, β).

The condition of coanorthotomy for these conics is, by § 19, *Cor.*,

$$f\alpha+g\beta+h=1.$$

Hence the linear relation is deduced, as in former investigations,

$$\alpha(Lx-fS^{\frac{1}{2}})+\beta(My-gS^{\frac{1}{2}})+N-hS^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots(1).$$

Let the dirigent conic (F), on which (α, β) is constrained to lie, be

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The variation of (1), subject to (2), determines the binodal quartic

$$a^2(Lx-fS^{\frac{1}{2}})^2+b^2(My-gS^{\frac{1}{2}})^2=(N-hS^{\frac{1}{2}})^2,$$

or, when expanded,

$$a^2L^2x^2+b^2M^2y^2-N^2+(a^2f^2+b^2g^2-h^2)S=2S^{\frac{1}{2}}(a^2fLx+b^2gMy-hN).$$

Two conclusions may be drawn:

(1) The four points of the quadruple contact are the intersections of S with the conic

$$a^2L^2x^2+b^2M^2y^2-N^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

They are, therefore, points of S, at which the tangents also touch F.

(2) The line of nodes is in this case the polar of the centre of contact of any (J) conic with respect to (3), which itself varies with (F) the dirigent.

Another dirigent (F_1) must have the form in tangential coordinates

$$\left(a^2+\frac{\kappa}{L}\right)\xi^2+\left(b^2+\frac{\kappa}{M}\right)\eta^2-1+\frac{\kappa}{N}=0,$$

since F and F_1 are contangential with S. If a', b' denote its semi-axes,

$$a'^2\left(1-\frac{\kappa}{N}\right)=a^2+\frac{\kappa}{L}, \quad b'^2\left(1-\frac{\kappa}{N}\right)=b^2+\frac{\kappa}{M}.$$

Hence $$N+La^2 : N+La'^2 :: N+Mb^2 : N+Mb'^2.$$

It appears, therefore, that the coordinates of the points of quadruple contact are

$$x^2\left(a^2L-b^2M\right)=\frac{N}{L}\left(N+b^2M\right),\quad y^2\left(a^2L-b^2M\right)=-\frac{N}{M}\left(N+a^2L\right),$$

and they are independent of any particular dirigent, as they should be.

30. To generate a binodal quartic, which shall have triple contact with a given conic S, one contact being of the third order.

Since the points of contact, by § 23, are those points in S at which the tangents also touch the dirigent conic F, two of these tangents must unite in this case, and S and F must touch each other.

Let $S\equiv ux^2+vy^2+2w'xy+2v'x=0$, $F\equiv ax^2+by^2+2cxy+2v'x=0$.

Assume the fixed conic

$$J\equiv fx+gy+h-S^{\frac{1}{2}}.$$

The generating conic, whose centre of contact is (α, β), is

$$G\equiv \alpha S_1+\beta S_2+S_3-S^{\frac{1}{2}}=0,$$

if $$2S_1=\frac{dS}{dx},\quad 2S_2=\frac{dS}{dy},\quad S_3=v'x.$$

By § 19 *Cor.*, the condition of coanorthotomy for G and J is

$$f\alpha+g\beta+h=1.$$

Hence the linear relation is deduced,

$$\alpha\left(S_1-fS^{\frac{1}{2}}\right)+\beta\left(S_2-gS^{\frac{1}{2}}\right)+S_3-hS^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots (1).$$

Also, since (α, β) must lie on F,

$$a\alpha^2+b\beta^2+2c\alpha\beta+2v'\alpha=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The variation of (1), subject to the condition (2), determines the binodal

$$\left[v'\left(S_2-gS^{\frac{1}{2}}\right)-c\left(S_3-hS^{\frac{1}{2}}\right)\right]^2$$

$$+2bv'\left(S_1-fS^{\frac{1}{2}}\right)\left(S_3-hS^{\frac{1}{2}}\right)-ab\left(S_3-hS^{\frac{1}{2}}\right)^2=0.$$

At the points of contact with S, $S=0$,

and $$\left[vy+(w'-c)x\right]^2+2bx\left(ux+w'y+v'\right)-abx^2=0,$$

or $$x^2\left[(w'-c)^2+b(2u-a)\right]+2xy\left[bw'+v(w'-c)\right]+v^2y^2+2bv'x=0.$$

Again, combine with $(S=0)$,

$$x^2\left[(w'-c)^2+b(u-a)\right]+2vxy(w'-c)+v(v-b)y^2=0.$$

These two lines determine, by their intersection with S, the two

points at which the binodal has contact of the first order with S. At the origin the contact is of the third order.

It may be simply seen that, at the first-named two points, S and F have common tangents. The tangents to the two conics are

$$x(ua+w'\beta+v')+y(w'a+v'\beta)+v'a=0,$$

$$x(aa_1+c\beta_1+v')+y(ca_1+b\beta_1)+v'a_1=0.$$

Equate their coefficients,

$$S_1 \propto aa_1+c\beta_1+v', \quad S^2 \propto ca_1+b\beta_1, \quad S_3 \propto v'a_1.$$

When these values of a_1, β_1, γ_1 are substituted in F,

$$(v'S_3-cS_2)^2+2bv'S_1S_3-abS_3^2=0,$$

which determines the points of contact in the binodal.

31. To generate a binodal quartic, which shall have double contact with a given conic, one contact being of the fifth order.

Assume S as in § 30, but write v for b in F; then, at the origin, S and F have contact of the second order, and therefore the binodal generated, as in § 30, has, at the origin, contact of the fifth order with S.

32. If both $b=v$, and $c=w'$, *i.e.*, if S and F have a contact of the third order, the binodal quartic thus generated has with S a contact of the seventh order, or 8-pointic. Its equation has the form

$$x^2+rS+u_1S^4=0,$$

if u_1 denote the line of nodes.

33. A second set of coanorthotomic Jacobians is suggested by coorthotomic Jacobians, which are employed to generate sphero-quadrics, when the polar triangle is used for reference (*Proceedings of Lond. Math. Soc.*, Vol. XVI., p. 117).

If S, K, u, ... u', ... U, ... U' ... have the same meaning as in § 19, the following conics, which have double contact with S, are coanorthotomic, and any one is the Jacobian of the other three.

$$\surd K\,U'a = \surd(U'V'W')\surd S \quad \ldots\ldots\ldots\ldots\ldots\ldots (J_1),$$

$$\surd K\,V'\beta = \surd(U'V'W')\surd S \quad \ldots\ldots\ldots\ldots\ldots\ldots (J_2),$$

$$\surd K\,W'\beta = \surd(U'V'W')\surd S \quad \ldots\ldots\ldots\ldots\ldots\ldots (J_3),$$

$$\surd K\,(v'w'a+w'u'\beta+u'v'\gamma) = \surd(U'V'W')\surd S \quad \ldots\ldots\ldots (J).$$

Their centres of contact are (U, W', V'), (W', V, U'), (V', U', W), $(V'W', W'U', U'V')$ respectively (the last being the same as for J in § 20), and poles, with respect to any other Jacobian of the system, of a common chord of such a pair.

Assume the generating conic to be

$$\lambda\alpha+\mu\beta+\nu\gamma = S^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (G).$$

The condition of coanorthotomy with (J) is, by § 19,

$$\surd K\ \surd(U'V'W') = \lambda V'W'+\mu W'U'+\nu U'V',$$

since $$Uv'w'+W'w'u'+V'u'v' = V'W'.$$

Hence $$\frac{\lambda}{U'}\left\{U'\alpha-\frac{\surd S}{\surd K}\surd(U'V'W')\right\}+\ldots = 0,$$

or $$\frac{\lambda J_1}{U'}+\frac{\mu J_2}{V'}+\frac{\nu J_3}{W'} = 0.$$

The Jacobian (J) may be presented in the form

$$\frac{u'}{U'}(u\alpha+w'\beta+v'\gamma)^2+\frac{v'}{V'}(w'\alpha+v\beta+u'\gamma)^2+\frac{w'}{W'}(v'\alpha+u'\beta+w\gamma)^2 = 0 \ldots\ldots\ldots\ldots (J).$$

The equation to the corresponding dirigent conic (F), on which the centre of contact of the generator (G) must lie, must be, when it is referred to the same autopolar triangle,

$$l(u\alpha+w'\beta+v'\gamma)^2+m(w'\alpha+v\beta+u'\gamma)^2+n(v'\alpha+u'\beta+w\gamma)^2 = 0 \ldots\ldots (F).$$

Now the coordinates of the centre of contact of (G) vary as

$$U\lambda+W'\mu+V'\nu,\quad W'\lambda+V\mu+U'\nu,\quad V'\lambda+U'\mu+W\nu.$$

Since this is constrained to lie on (F),

$$l\lambda^2+m\mu^2+n\nu^2 = 0.$$

The variables are subject to the other condition,

$$\frac{\lambda J_1}{U'}+\frac{\mu J_2}{V'}+\frac{\nu J_3}{W'} = 0.$$

By differentiating, we obtain the equation to the binodal quartic, as the envelope of the generator (G),

$$\frac{J_1^2}{lU'^2}+\frac{J_2^2}{mV'^2}+\frac{J_3^2}{nW'^2} = 0.$$

When developed, if $\quad r\surd K = \surd(U'V'W'),$

$$\frac{1}{l}(\alpha-\frac{1}{U'}rS^{\frac{1}{2}})^2+\frac{1}{m}(\beta-\frac{1}{V'}rS^{\frac{1}{2}})^2+\frac{1}{n}(\gamma-\frac{1}{W'}rS^{\frac{1}{2}}).$$

The four points in which the given conic (S) intersects

$$\frac{1}{l}\alpha^2+\frac{1}{m}\beta^2+\frac{1}{n}\gamma^2 = 0$$

are the points in which S touches the binodal quartic.

It may be shown, as in §§ 20, 25, that a binodal quartic can be generated in four ways, which shall touch a given conic (S) in four assigned points, at which the tangents at S and F are common.

34. The theorems, established in this memoir, are equally applicable to Spherical Geometry, and their proofs identical, if the coordinates α, β, γ be interpreted as the sines of perpendicular arcs, and the Cartesian coordinates x, y as tangents of intercepted arcs in Gudermann's system.

On the Genesis of Class-quartics with two Bitangents from Conics.

35. This genesis is virtually determined in the preceding pages, and the analytical processes therein employed are identical, if the coordinates be interpreted as tangential.

The definitions of coanorthotomy in the two families of binodal quartics should be thus dualised:—

Def. 1.—Two conics P, Q have each the same pair of tangents as a third conic S; a certain definite tangency exists between a second pair of common tangents of P and Q, when the polar of the intersection of the first pair, with respect to P or Q, is also the polar of the intersection of the second pair with respect to Q or P respectively.

Def. 2.—Two conics P, Q have each double contact with a third conic S; a certain definite tangency exists between P and Q, when the chord of contact for P or Q is also the polar of the intersection of a pair of common tangents of P and Q with respect to Q or P respectively.

Thursday, June 12th, 1890.

J. J. Walker, Esq., F.R.S., President, in the Chair.

Mr. R. A. Sampson was admitted into the Society.

The President announced that the Council had unanimously awarded the De Morgan Gold Medal to Lord Rayleigh for his papers and other writings on Mathematical Physics.

The following papers were read:—

Further Notes on Simplicissima: Mr. W. J. C. Sharp.

Rotatory Polarization: Dr. J. Larmor.

Some Properties of Conormal Points on a Parabola: Mr. Tucker.

Prof. Greenhill communicated papers—

On the Expression of the Square Root of a Quartic as a Continued Fraction: Prof. G. B. Mathews; and

On Modular Equations: Mr. R. Russell.

The President gave a brief sketch of a paper—

On certain Concomitants of a System of Conics and Quadrics, and on the Calculation of the Covariant S of the Ternary Quartic: Mr. A. R. Johnson.

The following presents were received:—

"Educational Times," for June.

Royal Irish Academy: Cunningham Memoirs—"The Red Stars, Observations and Catalogue," by J. Birmingham; New edition, by Rev. T. E. Espin; 4to, Dublin, 1890.

"An Elementary Treatise upon the Method of Least Squares, with Numerical Examples of its Applications," by G. C. Comstock, 8vo; Boston, 1890.

"Bulletin des Sciences Mathématiques," Index to Vol. XIII., Tome XIV.; May, 1890.

"Bulletin de la Société Mathématique de France," Tome XVIII., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stücke 4 and 5.

"Acta Mathematica," XIV., 2; 1890.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 6 and 7; 1890.

"Annali di Matematica," Tomo XVIII., Fasc. 1; 1890.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," 21 Heft, 1889; München, 1890.

"Berichte über die Verhandlungen der Königlich-Sachsischen Gesellschaft der Wissenschaften zu Leipzig," 1889, II., III., IV.; Register zu den Jahrgängen, 1846–1885.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," XXXIV., 3 and 4; 1889.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 105 and 106.

"Rendiconti del Circolo Matematico di Palermo," Tomo IV., Fasc. III. and IV.; 1890.

"Œuvres de Fourier," publiées par les soins de M. Gaston Darboux, Tome 2, 4to; Paris, 1890.

"Ueber den Schwerpunkt des Menschlichen Körpers," by W. Braune and O. Fischer, 8vo; Leipzig, 1889.

"Die Formentwickelung des Menschlichen Vorderhirns vom Ende des ersten bis zum Beginn des dritten Monats," by W. His, 8vo; Leipzig, 1889.

"Zahl und Vertheilung der markhaltigen Fasern im Froschrückenmark," by Justus Gaule, 8vo; Leipzig, 1889.

"Leçons sur la Théorie Générale des Surfaces," by Gaston Darboux, III. Partie, Fasc. 1, 8vo; Paris, 1890.

"Annales de la Faculté des Sciences de Toulouse pour les Sciences Mathématiques et les Sciences Physiques," Tome III., 1–4 fascicules; 1889.

On Simplicissima in Space of n Dimensions. (Third Paper.)

By W. J. Curran Sharp.

[*Read June 12th*, 1890.]

SUMMARY OF THE PAPER.

Addenda to Arts. X., XIII., XXVII., XXIX., and XXXVI.

Articles.

XXXVII. Content of subsidiary simplicissima in terms of the coordinates of the vertices.

XXXVIII. The angle between two right lines. Condition of perpendicularity. Shortest distance between two right lines. Principal axes of a quadric.

XXXIX. Content of the simplicissimum vertices at any point and all but one of its projections on the faces of the simplicissimum of reference; of the pedal simplicissimum. Condition that this should vanish.

XL. Interpretation of various equations by the help of XXXIX., especially $\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\ldots=0$. The poles of infinity. All lines through these are such that the sum of the reciprocals of the intercepts is zero.

XLI. The identical relations between the products of complementary simplicissima of n vertices, situated at $2n$ points in space of $n-1$ dimensions. Triangles having their vertices at six or five points in a plane. Tetrahedron with vertices at eight, seven, or six points. Various classes of anharmonic systems. All up to the $n-1^{\text{th}}$ are applicable in the Geometry of space of n dimensions.

XLII. Analogue to harmonical progression. Complete system of 2^{n-1} points harmonically related to n points. Properties of such systems.

XLIII. Quadrics intersecting in two linear loci, one of which is infinity, are similar and similarly situated, and have two centres of similitude.

Articles.

XLIV. Any two quadrics have a common self-conjugate simplicissimum. Its vertices. When these become indeterminate.

XLV. The value of the discriminant of

$$U_{11}\lambda^2+U_{22}\mu^2+\ldots+2U_{12}\lambda\mu+\ldots.$$

XLVI. The invariants, contravariants, and covariants of two quadrics.

Addenda to Former Papers.

(Vol. XVIII., pp. 325–59; Vol. XIX., pp. 423–82.)

X. If $\quad \Sigma\equiv S-(A_1\lambda+A_2\mu+A_3\nu+\ldots)(\lambda+\mu+\nu\ldots),$

then $$\Sigma-\left(\lambda'\frac{d\Sigma}{d\lambda}+\mu'\frac{d\Sigma}{d\mu}+\nu'\frac{d\Sigma}{d\nu}+\ldots\right)(\lambda+\mu+\nu+\ldots)=0\ldots\ldots(1)$$

represents the spheric with centre at $(\lambda', \mu', \nu'\ldots)$, and which cuts $\Sigma=0$ orthogonally. For at the centre of (1),

$$\frac{d\Sigma}{d\lambda}-\frac{d\Sigma'}{d\lambda'}=\frac{d\Sigma}{d\mu}-\frac{d\Sigma'}{d\mu'}=\frac{d\Sigma}{d\nu}-\frac{d\Sigma'}{d\nu'}=\ldots,$$

and the radical locus of (1) and $\Sigma=0$ is the polar of the centre of each with respect to the other.

XIII. The proof of this proposition, which is equally true if instead of spherics we say similar circumscribed quadrics (Art. XLIII.), is incomplete as it stands, as it is not proved that the common point on the radical loci lies upon the spherics. This is easily proved by multiplying the equations to the radical loci by $\lambda, \mu, \nu, \ldots$, respectively, and adding, when the result is $S=0$.

XXVII. The central axis is at right angles to the common radical locus of the spherics

$$A_1\lambda+A_2\mu+A_3\nu+\ldots=0,$$

for the equations to the axis may be written

$$\frac{\lambda-\dfrac{V}{n+1}}{\dfrac{1}{A_1}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\frac{\mu-\dfrac{V}{n+1}}{\dfrac{1}{A_2}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\frac{\nu-\dfrac{V}{n+1}}{\dfrac{1}{A_3}-\dfrac{1}{n+1}\Sigma\dfrac{1}{A_1}}=\&c.,$$

for which values of $a, b, c \ldots$ (Art. XVIII.),

$$\frac{d}{da} S(a, b, c \ldots) = n-1-\frac{1}{n+1} \Sigma \frac{1}{A_1} \Sigma A_1 + \frac{2}{n+1} A_1 \Sigma \frac{1}{A_1},$$

$$\frac{d}{db} S(a, b, c \ldots) = n-1-\frac{1}{n+1} \Sigma \frac{1}{A_1} \Sigma A_1 + \frac{2}{n+1} A_2 \Sigma \frac{1}{A_1},$$

&c. &c.

Therefore

$$\left\| \begin{matrix} \frac{d}{da} S(a, b, c \ldots), & \frac{d}{db} S(a, b, c \ldots), & \frac{d}{dc} S(a, b, c \ldots), & \ldots \\ A_1, & A_2, & A_3, & \ldots \\ 1, & 1, & 1, & \ldots \end{matrix} \right\| = 0,$$

therefore the axis is at right angles to the radical locus, and the centres of all the spherics lie upon it.

The shortest distance between two non-intersecting edges (p, q) and (r, s) of a rectangular simplicissimum is the square root of

$$\frac{A_p A_q}{A_p + A_q} + \frac{A_r A_s}{A_r + A_s}.$$

Let $(\lambda_1, \mu_1, 0, 0)$, $(0, 0, \pi_3, \rho_3, \ldots)$ be the points where the shortest distance between (1.2) and (3.4) meets these lines respectively; then, if d be this distance,

$$\begin{aligned} V^2 d^2 &= -S_1 - S_2 + \left(\lambda_1 \frac{dS_2}{d\lambda_2} + \mu_1 \frac{dS_2}{d\mu_2} + \nu_1 \frac{dS_2}{d\nu_2} - \ldots\right) \\ &= -\lambda_1 \mu_1 (A_1 + A_2) - \pi_3 \rho_3 (A_3 + A_4) \\ &\quad + \lambda_1 \{\pi_3 (A_1 + A_3) + \rho_3 (A_1 + A_4)\} \\ &\quad + \mu_1 \{\pi_3 (A_2 + A_3) + \rho_3 (A_2 + A_4)\} \\ &= -\lambda_1 (V - \lambda_1)(A_1 + A_2) - \pi_3 (V - \pi_3)(A_3 + A_4) \\ &\quad + \lambda_1 \{\pi_3 (A_1 + A_3) + (V - \pi_3)(A_1 + A_4)\} \\ &\quad + (V - \lambda_1) \{\pi_3 (A_2 + A_3) + (V - \pi_3)(A_2 + A_4)\}; \end{aligned}$$

and, by differentiation,

$$0 = -2VA_3 + 2\lambda_1 (A_1 + A_3), \quad 0 = -2VA_4 + 2\pi_3 (A_3 + A_4),$$

and

$$d^2 = \frac{A_1 A_2}{A_1 + A_2} + \frac{A_3 A_4}{A_3 + A_4}.$$

XXIX. If $\Sigma' = 0$ be the bisecting spheric, $\Sigma'' = 0$ the spheric through the centres, and $\Sigma''' = 0$ the orthogonal spheric, they are coaxal; for, from the equations as given,

$$\Sigma' + \Sigma''' \equiv 2S - 2\left\{\frac{\lambda'+\lambda'''}{2}\frac{dS}{d\lambda} + \frac{\mu'+\mu'''}{2}\frac{dS}{d\mu} + \dots\right\}\frac{\lambda+\mu+\dots}{V}$$
$$+ \frac{V^2r'^2 + V^2r'''^2 + S' + S'''}{V^2}(\lambda+\mu+\dots)^2$$
$$\equiv 2\left\{S - \left(\lambda''\frac{dS}{d\lambda} + \mu''\frac{dS}{d\mu} + \dots\right)\right\}\frac{\lambda+\mu+\dots}{V}$$
$$+ \frac{V^2r''^2 + S''}{V^2}(\lambda+\mu+\dots)^2 \equiv 2\Sigma''.$$

XXXVI. If the equation to a linear locus be put into the form

$$A_1\frac{dS}{d\lambda} + A_2\frac{dS}{d\mu} + A_3\frac{dS}{d\nu} + \dots = 0 \equiv L,$$

and
$$\lambda' : \mu' : \nu' \dots :: A_1 : A_2 : A_3,$$

[in other words, if $(\lambda'\mu'\nu' \dots)$ be the pole of the linear locus with respect to the circumspheric], the equation may be written

$$\lambda\frac{dS'}{d\lambda'} + \mu\frac{dS'}{d\mu'} + \nu\frac{dS'}{d\nu'} + \dots = 0,$$

the determinant of which is (Art. XII.), $(-2)^{n+1}(n!)^2 V^4 d^2$, where d is the distance from $(\lambda', \mu' \dots)$ to the circumcentre.

And hence the perpendicular from $(\lambda_1, \mu_1, \nu_1 \dots)$ upon $L = 0$

$$= \sqrt{\left\{\frac{1}{(\Sigma A_1)^2}\cdot\frac{(-2)^{n-1}(n!)^2}{(-2)^{n+1}(n!)^2}\cdot\frac{L_1^2}{V^2d^2}\right\}}$$
$$= \frac{1}{2\Sigma A_1}\cdot\frac{L_1}{Vd}.$$

Hence the perpendiculars from $(\lambda_1, \mu_1 \dots)$ upon

$$\frac{dS}{d\lambda} = 0, \quad \frac{dS}{d\mu} = 0, \text{ \&c.},$$

the tangents to the circumspheric at the vertices, are

$$\frac{1}{2VR}\cdot\frac{dS_1}{d\lambda_1}, \quad \frac{1}{2VR}\cdot\frac{dS_1}{d\mu_1}, \text{ \&c.},$$

and the coordinates in the system mentioned in Art. XXXVI. are

proportional to perpendiculars upon the faces of a simplicissimum and analogous to trilinear or quadruplanar coordinates.

[The following Articles are numbered in continuation of the former papers.]

XXXVII. Professor Sylvester's fundamental formula (Art. 1., Vol. XVIII., p. 325) for the squared content of a simplicissimum of $n+1$ vertices, viz.,

$$\frac{-1}{(-2)^n (1 \cdot 2 \ldots n)^2} \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \ldots \\ 1, & (2.1)^2, & 0, & (2.3)^2, & \ldots \\ 1, & (3.1)^2, & (3.2)^2, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix},$$

may be transformed so as to give the squared content of any simplicissimum in terms of the coordinates of the vertices.

For
$$V^2 (r.s)^2 = -S_r - S_s + \left(\lambda_r \frac{dS_s}{d\lambda_r} + \mu_r \frac{dS_s}{d\mu_s} + \ldots\right)$$
$$\equiv -S_r - S_s + S_{r.s}, \text{ say};$$

therefore the squared area of the triangle, the vertices of which are at

$$(\lambda_1, \mu_1, \nu_1, \ldots), (\lambda_2, \mu_2, \nu_2, \ldots), (\lambda_3, \mu_3, \nu_3, \ldots),$$

$$= \frac{-1}{16} \begin{vmatrix} 0, & 1, & 1, & 1, \\ 1, & 0, & \dfrac{-S_1 - S_2 + S_{1.2}}{V^2}, & \dfrac{-S_1 - S_3 + S_{1.3}}{V^2} \\ 1, & \dfrac{-S_2 - S_1 + S_{1.2}}{V^2}, & 0, & \dfrac{-S_2 - S_3 + S_{2.3}}{V^2} \\ 1, & \dfrac{-S_3 - S_1 + S_{1.3}}{V^2}, & \dfrac{-S_3 - S_2 + S_{2.3}}{V^2}, & 0, \end{vmatrix}$$

$$= \frac{-1}{16V^4} \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 2S_1, & S_{1.2}, & S_{1.3} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3} \\ 1, & S_{1.3}, & S_{2.3}, & 2S_3 \end{vmatrix}.$$

In the same way it may be shown that the squared content of the tetrahedron, the vertices of which are at

$$(\lambda_1, \mu_1, \ldots), (\lambda_2, \mu_2, \ldots), (\lambda_3, \mu_3, \ldots), (\lambda_4, \mu_4, \ldots),$$

$$= \frac{-1}{(-2)^3 (3!)^3} \frac{1}{V^6} \begin{vmatrix} 0, & 1, & 1, & 1, & 1, \\ 1, & 2S_1, & S_{1.2}, & S_{1.3}, & S_{1.4} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3}, & S_{2.4} \\ 1, & S_{1.3}, & S_{2.3}, & 2S_3, & S_{3.4} \\ 1, & S_{1.4}, & S_{2.4}, & S_{3.4}, & 2S_4 \end{vmatrix},$$

and more generally that the squared content of the simplicissimum of $p+1$ vertices, situated at

$$(\lambda_1, \mu_1, \ldots), (\lambda_2, \mu_2, \ldots) \ldots (\lambda_{p+1}, \mu_{p+1}, \ldots),$$

$$= \frac{-1}{(-2)^p (p!)^2} \frac{1}{V^{2p}} \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots & 1, \\ 1, & 2S_1, & S_{1.2}, & S_{1.3}, & \ldots & S_{1.p+1} \\ 1, & S_{1.2}, & 2S_2, & S_{2.3}, & \ldots & S_{2.p+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & S_{1.p+1}, & S_{2.p+1}, & S_{3.p+1} & \ldots & 2S_{p+1} \end{vmatrix}.$$

From this formula it appears that the locus of the $p+1^{\text{th}}$ vertex of a simplicissimum of $p+1$ vertices, the content of which is constant and the base (the other p vertices) fixed, is a locus of the second degree. The simplest examples of this class of loci are the circle (or spheric) and the right circular cylinder. Such loci pass through the intersection of the spherical locus at infinity, with the linear loci

$$\lambda \frac{dS_1}{d\lambda_1} + \mu \frac{dS_1}{d\mu_1} + \nu \frac{dS_1}{d\nu_1} + \ldots = 0,$$

$$\lambda \frac{dS_2}{d\lambda_2} + \mu \frac{dS_2}{d\mu_2} + \nu \frac{dS_2}{d\nu_2} + \ldots = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\lambda \frac{dS_p}{d\lambda_p} + \mu \frac{dS_p}{d\mu_p} + \nu \frac{dS_p}{d\nu_p} + \ldots = 0.$$

If $p = n$, in the formula of this article, the squared content of the simplicissimum vertices at

$$(\lambda_1, \mu_1 \ldots) (\lambda_2, \mu_2 \ldots) \ldots (\lambda_{n+1}, \mu_{n+1} \ldots)$$

$$= \frac{-1}{(-2)^n (n!)^2} \frac{1}{V^{2n}} \begin{vmatrix} 0, & 1, & 1, & \ldots\ldots & 1 \\ 1, & 2S_1, & S_{1.2}, & \ldots\ldots & S_{1.n+1} \\ 1, & S_{1.2}, & 2S_2 & \ldots\ldots & S_{2.n+1} \\ & \ldots & \ldots & \ldots & \ldots \\ 1, & S_{1.n+1}, & S_{2.n+1}, & \ldots & 2S_{n+1} \end{vmatrix}$$

$$= \frac{-1}{(-2)^n (n!)^2} \frac{1}{V^{2n+1}} \begin{vmatrix} 1, & 0, & 0, & \ldots\ldots & 0 \\ 1, & \lambda_1, & \mu_1, & \ldots\ldots & \tau_1 \\ 1, & \lambda_2, & \mu_2, & \ldots\ldots & \tau_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & \lambda_{n+1}, & \mu_{n+1}, & \ldots & \tau_{n+1} \end{vmatrix} \times \begin{vmatrix} 0, & 1, & 1, & \ldots\ldots & 1 \\ 1, & \frac{dS_1}{d\lambda_1}, & \frac{dS_1}{d\mu_1}, & \ldots & \frac{dS_1}{d\tau_1} \\ 1, & \frac{dS_2}{d\lambda_2}, & \frac{dS_2}{d\mu_2}, & \ldots & \frac{dS_2}{d\tau_2} \\ & \ldots & \ldots & \ldots & \ldots \\ 1, & \frac{dS_{n+1}}{d\lambda_{n+1}}, & \frac{dS_{n+1}}{d\mu_{n+1}}, & \ldots & \frac{dS_{n+1}}{d\tau_{n+1}} \end{vmatrix}$$

and (Art. VI.)

$$\frac{1}{V^n} \begin{vmatrix} 1, & 0, & 0, & \ldots & 0 \\ 1, & \lambda_1, & \mu_1, & \ldots & \tau_1 \\ 1, & \lambda_2, & \mu_2, & \ldots & \tau_2 \\ & \ldots & \ldots & \ldots & \ldots \\ 1, & \lambda_{n+1}, & \mu_{n+1}, & \ldots & \tau_{n+1} \end{vmatrix}$$

is equal to the content of the simplicissimum; therefore this is also

$$= \frac{-1}{(-2)^n (n!)^2 V^{n+1}} \begin{vmatrix} 0, & 1, & 1, & \ldots\ldots & 1 \\ 1, & \frac{dS_1}{d\lambda_1}, & \frac{dS_1}{d\mu_1}, & \ldots & \frac{dS_1}{d\tau_1} \\ 1, & \frac{dS_2}{d\lambda_2}, & \frac{dS_2}{d\mu_2}, & \ldots & \frac{dS_2}{d\tau_2} \\ & \ldots & \ldots & \ldots & \ldots \\ 1, & \frac{dS_{n+1}}{d\lambda_{n+1}}, & \frac{dS_{n+1}}{d\mu_{n+1}}, & \ldots & \frac{dS_{n+1}}{d\tau_{n+1}} \end{vmatrix},$$

which gives an expression in terms of the coordinates of Art. XXXVI.

XXXVIII. If $$\frac{\lambda - \lambda'}{a} = \frac{\mu - \mu'}{b} = \frac{\nu - \nu'}{c} = \ldots\ldots \quad \ldots\ldots\ldots(1),$$

where $$a + b + c + \ldots = 0,$$

and $$\frac{\lambda - \lambda''}{a} = \frac{\mu - \mu''}{b} = \frac{\nu - \nu''}{c} = \ldots\ldots \quad \ldots\ldots\ldots(2),$$

where $$a + b + c + \ldots = 0,$$

be two intersecting straight lines, and A the angle between them,

$$\cos A = \frac{\left(a \frac{d}{d\lambda} + b \frac{d}{d\mu} + c \frac{d}{d\nu} + \ldots\right) S(a, b, c \ldots)}{2 \cdot S(a, b, c \ldots) S(a, b, c \ldots)}.$$

The angle between the lines is the same as that between the parallels to them through $(V, 0, 0, \ldots)$ the vertex opposite $\lambda = 0$. Let those lines meet $\lambda = 0$ in $A_1(\lambda_1, \mu_1, \nu_1 \ldots)$ and $A_1'(\lambda_1', \mu_1', \nu_1' \ldots)$ respectively.

Now, at A_1,

$$\lambda_1 = 0, \quad \mu_1 = -\frac{b}{a} V, \quad \nu_1 = -\frac{c}{a} V, \text{ \&c.};$$

and at A_1',

$$\lambda_1' = 0, \quad \mu_1' = -\frac{b'}{a'} V, \quad \nu_1' = -\frac{c'}{a'} V, \text{ \&c.};$$

therefore
$$V^2 A A_1^2 = -S_1 + V \frac{dS_1}{d\lambda_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(X.)}$$

$$= -\frac{V^2}{a^2} S(a, b, c \ldots) + \frac{V^2}{a} \frac{d}{da} S(a, b, c \ldots)$$

$$- \frac{V^2}{a} \frac{d}{da} S(a, b, c \ldots)$$

$$= -\frac{V^2}{a^2} S(a, b, c \ldots),$$

and
$$A A_1^2 = -\frac{1}{a^2} S(a, b, c \ldots).$$

Similarly,
$$A A_1'^2 = -\frac{1}{a'^2} S(a', b', c' \ldots);$$

also,

$$V^2 A_1 A_1'^2 = -S_1 - S_1' + \left(\lambda_1' \frac{d}{d\lambda_1} + \mu_1' \frac{d}{d\mu_1} + \ldots\right) S_1$$

$$= -\frac{V^2}{a^2} S(a, b, c \ldots) + \frac{V^2}{a} \frac{d}{da} S(a, b, c \ldots)$$

$$- \frac{V^2}{a'^2} S(a', b', c' \ldots) + \frac{V^2}{a'} \frac{d}{da'} S(a', b', c' \ldots)$$

$$+ \frac{V^2}{aa'} \{(bc' + b'c)(2.3)^2 + (b'd + bd')(2.4)^2 + \ldots\}$$

$$= -\frac{V^2}{a^2} S(a, b, c \ldots) - \frac{V^2}{a'^2} S(a', b', c \ldots)$$

$$+ \frac{V^2}{aa'} \left\{ a' \frac{d}{da} + b' \frac{d}{db} + c' \frac{d}{dc} + \ldots \right\} S(a, b, c \ldots);$$

therefore

$$A A_1^2 + A A_1'^2 - A_1 A_1'^2 = -\frac{1}{aa'} \left\{ a' \frac{d}{da} + b' \frac{d}{db} + c' \frac{d}{dc} + \ldots \right\} S(a, b, c \ldots);$$

therefore

$$\cos\theta = \frac{AA_1^2+AA_1'^2-A_1A_1'^2}{2\,.\,AA_1\,.\,AA_1'} = \frac{\left\{a'\frac{d}{da}+b'\frac{d}{db}+\ldots\right\}S(a,b,c\ldots)}{2\sqrt{S(a,b,c\ldots)\,S(a',b',c'\ldots)}}.$$

Hence, if AA_1, BB_1, CC_1, &c. be parallel lines through the vertices A, B, C ..., meeting the opposite faces in A_1, B_1, C_1, &c. (Art. XVIII.),

$$\frac{1}{AA_1}+\frac{1}{BB_1}+\frac{1}{CC_1}+\ldots = \frac{a+b+c\ldots}{\sqrt{-S(a,b,c\ldots)}} = 0,$$

and

$$\frac{(1\,.\,2)^2}{AA_1\,.\,BB_1}+\frac{(2\,.\,3)^2}{BB_1\,.\,CC_1}+\frac{(1\,.\,3)^2}{AA\,.\,CC_1}+\&c. = \frac{S(a,b,c,\ldots)}{-S(a,b,c,\ldots)} = -1.$$

Also, if the lines be at right angles,

$$\left(a'\frac{d}{da}+b'\frac{d}{db}+c'\frac{d}{dc}+\ldots\right)S(a,b,c\ldots) = 0.$$

(Comp. Art. XXXV.) It appears too, that, if the line (2) lie upon the linear locus

$$A_1\lambda+A_2\mu+A_3\nu+\ldots = 0,$$

to which

$$\frac{\lambda-\lambda''}{a} = \frac{\mu-\mu''}{b} = \frac{\nu-\nu''}{c} = \ldots$$

is at right angles, the two lines are at right angles. For

$$A_1a'+A_2b'+A_3c'+\ldots = 0,$$

and

$$a'+b'+c'+\ldots = 0;$$

therefore

$$(A_2-A_1)\,b'+(A_3-A_1)\,c'+\ldots = 0,$$

and therefore (XVIII.)

$$\left(a'\frac{d}{da}+b'\frac{d}{db}+c'\frac{d}{dc}+\ldots\right)S(a,b,c\ldots) = 0,$$

and any right line at right angles to a linear locus is at right angles to all right lines in that locus, and conversely.

Again, the shortest distance between two non-intersecting straight lines is at right angles to each of them. Let these be

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots \equiv p \text{ say},$$

$$\frac{\lambda-\lambda''}{a'} = \frac{\mu-\mu''}{b'} = \frac{\nu-\nu''}{c'} = \ldots \equiv p' \text{ say},$$

and if $(\lambda_1, \mu_1 \ldots)$, $(\lambda_2, \mu_2 \ldots)$ be any points in these respectively, and d the distance between them,

$$\begin{aligned}
V^2 d^2 &= -S_1 - S_2 + \left(\lambda_2 \frac{dS_1}{d\lambda_1} + \mu_2 \frac{dS_1}{d\mu_1} + \ldots\right) \\
&= -S' - p\left(a\frac{dS'}{d\lambda'} + b\frac{dS'}{d\mu'} + c\frac{dS'}{d\nu'} + \ldots\right) \\
&\quad - p^2 S(a, b, c \ldots) - S'' - p'\left(a'\frac{dS''}{d\lambda''} + b'\frac{dS''}{d\mu''} + c'\frac{dS''}{d\nu''} + \ldots\right) \\
&\quad - p'^2 S(a', b', c' \ldots) + \left(\lambda''\frac{dS'}{d\lambda'} + \mu''\frac{dS'}{d\mu'} + \nu''\frac{dS'}{d\nu'} + \ldots\right) \\
&\quad + p\left(a\frac{dS''}{d\lambda''} + b\frac{dS''}{d\mu''} + c\frac{dS''}{d\nu''} + \ldots\right) \\
&\quad + p'\left(a'\frac{dS'}{d\lambda'} + b'\frac{dS'}{d\mu'} + c'\frac{dS'}{d\nu'} + \ldots\right) \\
&\quad + pp'\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \ldots\right) S(a, b, c \ldots),
\end{aligned}$$

which is to be a minimum by the variation of p and p', which are independent. This leads to two simple equations in p and p' which determine these and d^2 and $(\lambda_1, \mu_1 \ldots)$ $(\lambda_2, \mu_2 \ldots)$, and which can be transformed into

$$\left\{(\lambda_2 - \lambda_1)\frac{d}{da} + (\mu_2 - \mu_1)\frac{d}{db} + (\nu_2 - \nu_1)\frac{d}{dc} + \ldots\right\} S(a, b, c, \ldots) = 0,$$

and

$$\left\{(\lambda_1 - \lambda_2)\frac{d}{da'} + (\mu_1 - \mu_2)\frac{d}{db'} + (\nu_1 - \nu_2)\frac{d}{dc'} + \ldots\right\} S(a', b', c' \ldots) = 0.$$

So that the shortest distance is at right angles to both lines.

By the help of this article, the theory of the principal axes of a quadric locus may be completed. It has been shown (XXX.) that they are generally n in number; (XXXVI.) that they are mutually at right angles; and now, since they are by definition at right angles to the diametral loci which bisect chords parallel to them, it appears by the above the diametral locus which is conjugate to any principal axis contains all the others.

XXXIX. If $P(\lambda', \mu', \nu' \ldots)$ be any point, and A', B', $C' \ldots$ its projections on the faces of the simplicissimum of reference; $(PB'C'D' \ldots)$ the content of the simplicissimum with vertices at P, B', $C' \ldots$

$$= \frac{n^{\frac{1}{2}(n-1)}}{\{(n-1)!\}^{\frac{1}{2}}} \cdot \frac{\mu'}{V_2^2} \cdot \frac{\nu'}{V_3^2} \cdots V^{n-1},$$

and the pedal simplicissimum $A'B'C'\ldots$

$$= \frac{n^{2(n-1)}}{\{(n-1)!\}^2} \cdot \frac{\lambda'}{V_1^2} \cdot \frac{\mu'}{V_2^2} \cdot \frac{\nu'}{V_3^2} \ldots V^{n-1} \left\{ \frac{V_1^2}{\lambda'} + \frac{V_2^2}{\mu'} + \frac{V_3^2}{\nu'} + \ldots \right\}.$$

(Comp. Art. XXVI.)

Let
$$\frac{\lambda-\lambda'}{a_1} = \frac{\mu-\mu'}{b_1} = \frac{\nu-\nu'}{c_1} = \ldots \quad \ldots\ldots\ldots\ldots (1)$$
$$(a_1+b_1+c_1\ldots = 0),$$
$$\frac{\lambda-\lambda'}{a_2} = \frac{\mu-\mu'}{b_2} = \frac{\nu-\nu'}{c_2} = \ldots \quad \ldots\ldots\ldots\ldots (2)$$
$$(a_2+b_2+c_2+\ldots = 0),$$
$$\&c., \quad \&c.$$

be the equations to the lines PA', PB', PC', &c.; then (1) is at right angles to the linear locus

$$a_1 \frac{dS}{d\lambda} + b_1 \frac{dS}{d\mu} + c_1 \frac{dS}{d\nu} + \ldots = 0$$

through the circumcentre (Art. XXXVI.), or, writing S_1 for $S(a_1, b_1, c_1 \ldots)$, to

$$\lambda \frac{dS_1}{da_1} + \mu \frac{dS_1}{db_1} + \nu \frac{dS_1}{dc_1} + \ldots = 0,$$

and this is parallel to $\lambda = 0$; therefore

$$\frac{dS_1}{db_1} = \frac{dS_1}{dc_1} = \ldots \equiv q_1, \text{ say.}$$

And the equations to determine a_1, b_1, c_1, &c. are

$$a_1 + b_1 + \quad c_1 + \ldots = 0,$$
$$(2.1)^2 a_1 + \quad (2.3)^2 c_1 + \quad = q_1,$$
$$(3.1)^2 a_1 + (3.2)^2 \quad b_1 + \ldots = q_1,$$
$$\&c., \quad \&c.,$$

and

$$a_1 : b_1 : c_1 : \ldots :: \begin{vmatrix} 0, & 1, & 1, & 1 & \ldots \\ 1, & 0, & (2.3)^2, & (2.4)^2 & \ldots \\ 1, & (3.2)^2, & 0, & (3.4)^2 & \ldots \\ 1, & (4.2)^2, & (4.3)^2, & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$: \begin{vmatrix} 1, & 0, & 1, & 1 & \dots \\ (2.1)^2, & 1, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & 1, & 0, & (3.4)^2 & \dots \\ (4.1)^2, & 1, & (4.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} : \begin{vmatrix} 1, & 1, & 0, & 1 & \dots \\ (2.1)^2, & 0, & 1, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 1, & (3.4)^2 & \dots \\ (4.1)^2, & (4.2)^2, & 1, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$: \&c.;$$

and taking these as the values of a_1, b_1, c_1, &c.,

$$a_1 = -(-2)^{n-1}\{(n-1)!\}^2 V_1^2;$$

and

$$la_1 + mb_1 + nc_1 + \&c.$$

$$= -\begin{vmatrix} 0, & l, & m, & n & \dots \\ 0, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix};$$

therefore

$$\frac{dS_1}{db_1} = q_1 = \frac{dS_1}{dc_1} = \dots \equiv (2.1)^2 a_1 + (2.3)^2 c_1 + \dots$$

$$= -\begin{vmatrix} 0, & (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 1, & 1, & 1 & \dots \\ (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ (4.1)^2, & (4.2)^2, & (4.3)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

and

$$\frac{dS_1}{da_1} = (1.2)^2 b_1 + (1.3)^2 c_1 + \dots$$

$$= -\begin{vmatrix} 0, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 0, & 1, & 1, & 1 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 0, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} 1, & 1, & 1, & 1 & \dots \\ (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 & \dots \\ (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= -(-2)^n (n!)^2 V^2 + q_1,$$

and $$2S_1 = a_1 \frac{dS_1}{da_1} + b_1 \frac{dS_1}{db_1} + c_1 \frac{dS_1}{dc_1} + \dots$$

$$= -(-2)^n (n!)^2 V^2 \times \left[-(-2)^{n-1} \{(n-1)!\}^2 V_1^2 \right].$$

Thus, then, $a_1 = -(-2)^{n-1} \{(n-1)!\}^2 V_1^2$,

$$\frac{dS_1}{da_1} = -(-2)^n (n!)^2 V^2 + q_1,$$

$$S_1 = -(-2)^{2(n-1)} V^2 . V_1^2 (n!)^2 \{(n-1)!\}^2.$$

Similarly, $b_2 = -(-2)^{n-1} \{(n-1)!\}^2 V_2^2$,

$$\frac{dS_2}{db_2} = -(-2)^n (n!)^2 V^2 + q_2,$$

$$S_2 = -(-2)^{2(n-1)} (n!)^2 \{(n-1)!\}^2 V^2 V_2^2,$$

&c., &c.

Again, $$\begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & c_{n+1}, & \dots \end{vmatrix} \times \{-(-2)^n (n!)^2 V^2\}$$

$$= \begin{vmatrix} 1, & 0, & 0, & 0, & \dots \\ 0, & 1, & 0, & 0, & \dots \\ 0, & a_2, & b_2, & c_2, & \dots \\ 0, & a_3, & b_3, & c_3, & \dots \\ \dots & \dots & \dots & \dots & \\ 0, & a_{n+1}, & b_{n+1}, & c_{n+1}, & \dots \end{vmatrix} \times \begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (2.1)^2, & 0, & (2.3)^2, & \dots \\ 1, & (3.1)^2, & (3.2)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix}$$

$$= \begin{vmatrix} 0, & 1, & 0, & 0, & \dots \\ 1, & 0, & \frac{dS_2}{da_2}, & \frac{dS_3}{da_3}, & \dots \\ 1, & (2.1)^2, & \frac{dS_2}{db_2}, & \frac{dS_3}{db_3}, & \dots \\ 1, & (3.1)^2, & \frac{dS_2}{dc_2}, & \frac{dS_3}{dc_3}, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = - \begin{vmatrix} 1, & \frac{dS_2}{da_2}, & \frac{dS_3}{da_3}, & \dots \\ 1, & \frac{dS_2}{db_2}, & \frac{dS_3}{db_3}, & \dots \\ 1, & \frac{dS_2}{dc_2}, & \frac{dS_3}{dc_3}, & \dots \\ \dots & \dots & \dots & \end{vmatrix}$$

$$= - \begin{vmatrix} 1, & q_2, & q_3, & q_4, & \dots \\ 1, & \frac{dS_2}{db_2}, & q_3, & q_4, & \dots \\ 1, & q_2, & \frac{dS_3}{dc_3}, & q_4, & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = - \begin{vmatrix} 1, & q_2, & q_3, & \dots \\ 0, & \frac{dS_2}{db_2} - q_2, & 0, & \dots \\ 0, & 0, & \frac{dS_3}{dc_3} - q_3, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= - \left(\frac{dS_2}{db_2} - q_2\right)\left(\frac{dS_3}{dc_3} - q_3\right) \dots \left(\frac{dS_{n+1}}{dl_{n+1}} - q_{n+1}\right)$$

$$= -2^{n^2} (n!)^{2n} V^{2n};$$

$$\therefore \begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & (n+1), & \dots \end{vmatrix} = (-1)^n 2^{n(n-1)} (n!)^{2(n-1)} V^{2(n-1)}.$$

Now $$PA' = \frac{n\lambda'}{V_1}, \; PB' = \frac{n\mu'}{V_2}, \text{ &c.,}$$

and the direction ratios of these lines are the

$$a_1 : b_1 : c_1 : \dots ; \; a_2 : b_2 : c_2 \dots,$$

&c., &c., above.

And it was shown (Art. XX.) that $(PB'C'D' \dots)$

$$= \frac{1}{V^{n-1}} \frac{V^n}{\sqrt{(-1)^n S_2 S_3 \dots S_{n+1}}} \frac{n^n \mu' \nu' \dots}{V_2 . V_3} \begin{vmatrix} b_2, & c_2, & \dots \\ b_3, & c_3, & \dots \\ \dots & \dots & \dots \\ b_{n+1}, & c_{n+1}, & \dots \end{vmatrix}$$

$$= \frac{V}{(-1)^n 2^{n(n-1)} (n!)^n \{(n-1)!\}^n} \frac{1}{V_2 . V_3 \ldots V^n} \times \frac{n^n \mu' \nu'}{V_2 . V_3} \ldots$$

$$\times \{(-1)^n 2^{n(n-1)} (n!)^{2(n-1)} V^{2(n-1)}\}$$

$$= \frac{n^{2(n-1)}}{\{(n-1)!\}^2} \frac{\mu'}{V_2^2} \frac{\nu'}{V_3^2} \ldots V^{n-1},$$

with similar values for $(PA'C'D'\ldots)$, $(PA'B'D'\ldots)$, &c. And hence the pedal simplicissimum of P

$$= \frac{n^{2(n-1)} V^{n-1}}{\{(n-)1!\}^2} \frac{\lambda'}{V_1^2} \frac{\mu'}{V_2^2} \frac{\nu'}{V_3^2} \ldots \left\{ \frac{V_1^2}{\lambda'} + \frac{V_2^2}{\mu'} + \frac{V_3^2}{\nu'} + \text{&c.} \right\},$$

which vanishes if $\frac{V_1^2}{\lambda'} + \frac{V_2^2}{\mu'} + \frac{V_3^2}{\nu'} + \ldots = 0,$

in which case the projections of P lie on a linear locus (the Simson locus of Art. XXVI.).

XL. The value of $(A'B'C'D'\ldots)$ found in the last article leads to an interpretation of the equation

$$\frac{A}{\lambda} + \frac{B}{\mu} + \frac{C}{\nu} + \ldots = 0$$

to an n-ic locus circumscribed to the simplicissimum of reference, and having $n+1$ nodes at the vertices, the lines joining which lie wholly on the locus.

If points A'', B'', C'', &c. be taken on PA', PB', PC', &c., respectively, such that

$$PA'' = f . PA', \quad PB'' = g . PB', \quad PC'' = h . PC', \text{ &c.},$$

then, if $\frac{V_1^2}{f\lambda} + \frac{V_2^2}{g\mu} + \frac{V_3^2}{h\nu} + \ldots = 0$

(*i.e.*, the isogonal conjugate of $\frac{\lambda}{f} + \frac{\mu}{g} + \frac{\nu}{h} + \ldots = 0$)

be equivalent to $\frac{A}{\lambda} + \frac{B}{\mu} + \frac{C}{\nu} + \ldots = 0$ ..... (1);

i.e., if $f : g : h : \ldots :: \frac{V_1^2}{A} : \frac{V_2^2}{B} : \frac{V_3^2}{C} : \ldots ;$

then, for every position of P on (1), A'', B'', C'', &c. lie on a linear locus.

If the point lie upon the locus

$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\ldots=\frac{M}{\lambda\mu\nu}\ldots\left(\lambda+\mu+\nu\ldots\right)^2 \ldots\ldots\ldots\ldots (2)$$

(a locus concentric with the above, *i.e.*, having the same poles of infinity), the content of the simplicissimum whose vertices are at $A'', B'', C''+\ldots$ will be constant. (The above results are in analogy to those given for Conics in Questions 9816, 9817, 9818, and in the solution of Question 8177 in the *Reprint* from the *Educational Times*, Vol. XLIX.)

The linear polar of any point $(\lambda', \mu', \nu', \ldots)$ with respect to (1) is

$$\frac{\lambda}{\lambda'}\left(\frac{B}{\mu'}+\frac{C}{\nu'}+\ldots\right)+\frac{\mu}{\mu'}\left(\frac{A}{\lambda'}+\frac{C}{\nu'}+\ldots\right)+\&c.\ldots=0,$$

and hence the centres, the poles of infinity, are given by the equations

$$\frac{\lambda'}{A}\left(\frac{V}{n}-\lambda'\right)=\frac{\mu'}{B}\left(\frac{V}{n}-\mu'\right)=\frac{\nu'}{C}\left(\frac{V}{n}-\nu'\right)=\ldots.$$

If the point $(\lambda', \mu', \nu', \ldots)$ lie upon (1), this reduces to

$$\frac{A}{\lambda'^2}\lambda+\frac{B}{\mu'^2}\mu+\frac{C}{\nu'^2}\nu+\ldots=0 \ldots\ldots\ldots\ldots\ldots (3)$$

(compare Salmon, *Geometry of Three Dimensions*, p. 416), and therefore (Art. XII.), if $p_1, p_2, p_3, \ldots$ be the perpendiculars from the vertices of the simplicissimum of reference upon (3),

$$\frac{A}{\lambda'^2}:\frac{B}{\mu'^2}:\frac{C}{\nu'^2}:\ldots::p_1:p_2:p_3:\ldots,$$

and if $$A=V_1^2,\quad B=V_2^2,\quad C=V_3^2,\ \&c.,$$

so that the locus (1) is the isogonal conjugate of infinity, this implies that the perpendiculars from the vertices upon the tangent locus at any point are inversely proportional to the squares of the perpendiculars from the point of contact upon the corresponding faces of the simplicissimum (and hence this is true for the circumcircle of a triangle).

From (3), also, it follows that, if the linear locus

$$\alpha\lambda+\beta\mu+\gamma\nu+\ldots=0$$

touches (1),

$$(A\alpha)^{\frac{1}{2}}+(B\beta)^{\frac{1}{2}}+(C\gamma)^{\frac{1}{2}}+\ldots=0$$

(compare Salmon, *Geometry of Three Dimensions*, p. 416), and the reciprocal equation is of order 2^{n-1}.

If $(\lambda', \mu', \nu', \ldots)$ be any point, any line through it

$$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=\ldots\equiv p \text{ say}$$

$$(\text{where } a+b+c+\ldots=0)$$

will meet (1) at the points where p has the values given by the equation

$$A(\mu'+bp)(\nu'+cp)(\pi'+dp)\ldots+B(\lambda'+ap)(\nu'+cp)(\pi'+dp)\ldots$$
$$+C(\lambda'+ap)(\mu'+bp)(\pi'+dp)\ldots+\&c.=0;$$

or

$$\lambda'\mu'\nu'\ldots\Big[\Big(\frac{A}{\lambda'}+\frac{B}{\mu'}+\frac{C}{\nu'}+\ldots\Big)$$

$$+p\left\{\frac{A}{\lambda'}\Big(\frac{b}{\mu'}+\frac{c}{\nu'}+\ldots\Big)+\frac{B}{\mu'}\Big(\frac{a}{\lambda'}+\frac{c}{\nu'}+\ldots\Big)+\frac{C}{\nu'}\Big(\frac{a}{\lambda'}+\frac{b}{\mu'}+\ldots\Big)+\&c.\right\}$$

$$+p^2\left\{\frac{A}{\lambda'}\Big(\frac{bc}{\mu'\nu'}+\ldots\Big)+\frac{B}{\mu'}\Big(\frac{ac}{\lambda'\nu'}+\ldots\Big)+\frac{C}{\nu'}\Big(\frac{ab}{\lambda'\mu'}+\ldots\Big)+\&c.\right\}$$

$$+\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$+p^n\frac{abc\ldots}{\lambda'\mu'\nu'\ldots}\Big(\frac{A}{a}+\frac{B}{b}+\frac{C}{c}+\ldots\Big)\Big]=0,$$

and the line will lie wholly on the locus if the $n+1$ coefficients all vanish. This involves that $(\lambda'\mu'\nu'\ldots)$ should lie on (1), and that the eliminant of the other n coefficients equated to zero and of

$$a+b+c+\ldots=0$$

should vanish, which it must do since the edges lie on the locus. If $(\lambda'\mu'\nu'\ldots)$ be one of the poles of infinity, the coefficient of p in the above equation will vanish, and consequently the sum of the reciprocals of the intercepts made upon any line through one of them is zero. This is a generalisation of the property of the centre of a conic circumscribed to a triangle.

From the value of $(PB'C'D')$, found in the last article, it appears that

$$\mu\nu\ldots r=M(\lambda+\mu+\nu+\ldots+r)^n$$

represents a locus such that $(PB'C'D')$ is constant. The linear loci $\mu=0$, $\nu=0,\ldots$ are asymptotic to this, and their intersection, the vertex opposite to $\lambda=0$, is the pole of infinity. Also, any line wholly on the surface must be parallel to a face (not $\lambda=0$) of the simplicissimum.

Again,

$$\frac{A}{\lambda}+\frac{B}{\mu}=0, \text{ or } A\mu+B\lambda=0,$$

is a locus, such that $(PB'C'D'\ldots) : (PA'C'D'\ldots)$ is a constant ratio; and

$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}=0$$

a locus of the second degree which contains the intersections of

$$\lambda=0,\ \mu=0;\quad \mu=0,\ \nu=0;\quad \nu=0,\ \lambda=0,$$

is the locus of a point P such that

$$(PB'C'D'\ldots),\ (PA'C'D'\ldots),\ (PA'B'D'\ldots)$$

satisfy a linear relation.

Similarly,
$$\frac{A}{\lambda}+\frac{B}{\mu}+\frac{C}{\nu}+\frac{D}{\pi}=0$$

is a locus of the third order, such that, if P be any point in it,

$$(PB'C'D'\ldots),\ (PA'C'D'\ldots),\ (PA'B'D'\ldots),\ (PA'B'C'\ldots)$$

are connected by a linear relation. This locus passes through the intersections of any three of the linear loci

$$\lambda=0,\ \mu=0,\ \nu=0,\ \pi=0,$$

and the process of interpretation may be carried on indefinitely.

XLI. It has been shown (Art. XX.) that, if a pencil of $2n$ coterminous lines, in space of n dimensions, be cut by any linear locus, the products of the contents of pairs of complementary $(n-1)$-ary simplicissima, having their vertices at the points of intersection, must satisfy certain identical relations, which lead to relations among the anharmonic ratios of the pencil. These may be obtained as follows :—

All the $(n-1)^{\text{th}}$ minors of

$$\begin{vmatrix} b_1, & b_2, & b_3, & \ldots\ldots & b_{2n} \\ c_1, & c_2, & c_3, & \ldots\ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_1, & l_2, & l_3, & \ldots\ldots & l_{2n} \end{vmatrix}$$

$$\begin{vmatrix} b_1, & b_2, & b_3, & \ldots\ldots & b_{2n} \\ c_1, & c_2, & c_3, & \ldots\ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_1, & l_2, & l_3, & \ldots\ldots & l_{2n} \end{vmatrix}$$

must vanish. And therefore, writing $(p, q, r, \ldots t)$ for

$$\begin{vmatrix} b_p, & b_q, & b_r, & \ldots\ldots & b_t \\ c_p, & c_q, & c_r, & \ldots\ldots & c_t \\ \ldots & \ldots & \ldots & & \ldots \\ l_p, & l_q, & l_r, & \ldots\ldots & l_t \end{vmatrix},$$

$$b_n(n+1, n+2, \ldots 2n-1, 2n) + (-1)^n b_{n+1}(n+2, n+3, \ldots 2n, n) + b_{n+2}(n+3, n+4, \ldots n, n+1) + (-1)^n b_{n+3}(n+4, n+5, \ldots n+1, n+2) + \&c. = 0,$$

$$c_n(n+1, n+2, \ldots 2n-1, 2n) + (-1)^n c_{n+1}(n+2, n+3, \ldots 2n, n) + c_{n+2}(n+3, n+4, \ldots n, n+1) + (-1)^n c_{n+3}(n+4, n+5, \ldots n+1, n+2) + \&c. = 0,$$

&c. &c. &c.

Now, if these equations be multiplied respectively by the first minors of

$$\begin{vmatrix} b_n, & c_n, & \ldots\ldots & l_n \\ b_1, & c_1, & \ldots\ldots & l_1 \\ b_2, & c_2, & \ldots\ldots & l_2 \\ \ldots & \ldots & \ldots & \ldots \\ b_{n-1}, & c_{n-1}, & \ldots\ldots & l_{n-1} \end{vmatrix},$$

and the results added together,

$$(1, 2, \ldots n-1, n)(n+1, n+2, \ldots 2n-1, 2n) + (-1)^n (1, 2, \ldots n-1, n+1)(n+2, n+3, \ldots 2n, n) + (1, 2, \ldots n-1, n+2)(n+3, n+4, \ldots n, n+1) + (-1)^n (1, 2, \ldots n-1, n+3)(n+4, n+5, \ldots n+1, n+2) + \&c. = 0,$$

which is the general type of the identical relations required.

If $n = 2$, and 1, 2, 3, 4 denote four points in a straight line, the formula gives

$$(1, 2)(3, 4) + (1, 3)(4, 2) + (1, 4)(2, 3) = 0,$$

the well-known relation.

In general, the number of relations will be equal to the number of combinations of $2n$ things, $n-1$ together, *i.e.*,

$$\frac{2n!}{(n+1)!\,(n-1)!},$$

and therefore fifteen for tridimensional space. Thus, if 1, 2, 3, 4, 5, 6 denote any six points in a plane, and (1, 2, 3) the area of the triangle whose vertices are at 1, 2, and 3, taken in that order,

$$(1, 2, 3)(4, 5, 6) - (1, 2, 4)(5, 6, 3) + (1, 2, 5)(6, 3, 4) - (1, 2, 6)(3, 4, 5) = 0 \equiv \text{I.},$$

$$(1, 3, 4)(2, 5, 6) - (1, 3, 2)(5, 6, 4) + (1, 3, 5)(6, 4, 2) - (1, 3, 6)(4, 2, 5) = 0 \equiv \text{II.},$$

$$(1, 4, 5)(6, 2, 3) - (1, 4, 6)(2, 3, 5) + (1, 4, 2)(3, 5, 6) - (1, 4, 3)(5, 6, 2) = 0 \equiv \text{III.},$$

$$(1, 5, 6)(2, 3, 4) - (1, 5, 2)(3, 4, 6) + (1, 5, 3)(4, 6, 2) - (1, 5, 4)(6, 2, 3) = 0 \equiv \text{IV.},$$

$$(1, 6, 2)(3, 4, 5) - (1, 6, 3)(4, 5, 2) + (1, 6, 4)(5, 2, 3) - (1, 6, 5)(2, 3, 4) = 0 \equiv \text{V.},$$

$$(2, 3, 4)(6, 5, 1) - (2, 3, 6)(5, 1, 4) + (2, 3, 5)(1, 4, 6) - (2, 3, 1)(4, 6, 5) = 0 \equiv \text{VI.},$$

$$(2, 4, 5)(6, 3, 1) - (2, 4, 6)(3, 1, 5) + (2, 4, 3)(1, 5, 6) - (2, 4, 1)(5, 6, 3) = 0 \equiv \text{VII.},$$

$$(2, 5, 6)(4, 3, 1) - (2, 5, 4)(3, 1, 6) + (2, 5, 3)(1, 6, 4) - (2, 5, 1)(6, 4, 3) = 0 \equiv \text{VIII.},$$

$$(2, 6, 1)(5, 4, 3) - (2, 6, 5)(4, 3, 1) + (2, 6, 4)(3, 1, 5) - (2, 6, 3)(1, 5, 4) = 0 \equiv \text{IX.},$$

$$(3, 4, 5)(6, 1, 2) - (3, 4, 6)(1, 2, 5) + (3, 4, 1)(2, 5, 6) - (3, 4, 2)(5, 6, 1) = 0 \equiv \text{X.},$$

$$(3, 5, 6)(1, 2, 4) - (3, 5, 1)(2, 4, 6) + (3, 5, 2)(4, 6, 1) - (3, 5, 4)(6, 1, 2) = 0 \equiv \text{XI.},$$

$$(3, 6, 1)(2, 4, 5) - (3, 6, 2)(4, 5, 1) + (3, 6, 4)(5, 1, 2) - (3, 6, 5)(1, 2, 4) = 0 \equiv \text{XII.},$$

$$(4, 5, 6)(3, 2, 1) - (4, 5, 3)(2, 1, 6) + (4, 5, 2)(1, 6, 3) - (4, 5, 1)(6, 3, 2) = 0 \equiv \text{XIII.},$$

$$(4, 6, 1)(5, 3, 2) - (4, 6, 5)(3, 2, 1) + (4, 6, 3)(2, 1, 5) - (4, 6, 2)(1, 5, 3) = 0 \equiv \text{XIV.},$$

$$(5, 6, 1)(2, 3, 4) - (5, 6, 2)(3, 4, 1) + (5, 6, 3)(4, 1, 2) - (5, 6, 4)(1, 2, 3) = 0 \equiv \text{XV.};$$

and, denoting the sinisters of these equations by the Roman figures to which they are put equal,

$$\begin{array}{lll} \text{I.} + \text{X.} + \text{XV.} = 0, & \text{I.} + \text{XI.} + \text{XIV.} = 0, & \text{I.} + \text{XII.} + \text{XIII.} = 0, \\ \text{II.} + \text{VII.} + \text{XV.} = 0, & \text{II.} + \text{VIII.} + \text{XIV.} = 0, & \text{II.} + \text{IX.} + \text{XIII.} = 0, \\ \text{III.} + \text{VI.} + \text{XV.} = 0, & \text{III.} + \text{VIII.} + \text{XII.} = 0, & \text{III.} + \text{IX.} + \text{XI.} = 0, \\ \text{IV.} + \text{VI.} + \text{XIV.} = 0, & \text{IV.} + \text{VII.} + \text{XII.} = 0, & \text{IV.} + \text{IX.} + \text{X.} = 0, \\ \text{V.} + \text{VI.} + \text{XIII.} = 0, & \text{V.} + \text{VII.} + \text{XI.} = 0, & \text{V.} + \text{VIII.} + \text{X.} = 0. \end{array}$$

It is easy, by the help of these identities, to show that all the relations may be derived from five, but I have not succeeded in deriving them from four, as the theory (Art. XIX.) seems to require. This is probably due to my having overlooked some relation (probably not linear) between I., II., &c.

The above relations hold for any six points in a plane. If two points, say 5 and 6, coincide, those of the fifteen relations which do not become identically true, reduce to

$$(1, 2, 5)(3, 4, 5) - (1, 3, 5)(2, 4, 5) + (1, 4, 5)(2, 3, 5) = 0.$$

Hence, if 1, 2, 3, 4, 5 be any five points in a plane, the following relations hold among the triangles whose vertices are at these points :—

$$(1, 2, 5)(3, 4, 5) - (1, 3, 5)(2, 4, 5) + (1, 4, 5)(2, 3, 5) = 0,$$
$$(1, 2, 4)(3, 5, 4) - (1, 3, 4)(2, 5, 4) + (1, 5, 4)(2, 3, 4) = 0,$$
$$(1, 2, 3)(4, 5, 3) - (1, 4, 3)(2, 5, 3) + (1, 5, 3)(2, 4, 3) = 0,$$
$$(1, 3, 2)(4, 5, 2) - (1, 4, 2)(3, 5, 2) + (1, 5, 2)(3, 4, 2) = 0,$$
$$(2, 3, 1)(4, 5, 1) - (2, 4, 1)(3, 5, 1) + (2, 5, 1)(3, 4, 1) = 0.$$

If one of the anharmonic ratios be unity, others are so also; *e.g.*, if

$$(1, 2, 3)(4, 5, 6) = (1, 2, 4)(5, 6, 3),$$

from I., $(1, 2, 5)(6, 3, 4) = (1, 2, 6)(3, 4, 5)$,

and from XV., $(1, 5, 6)(2, 3, 4) = (2, 5, 6)(1, 3, 5)$

and the lines 12, 34, and 56 are concurrent.

If another independent ratio be unity also, say

$$(1, 3, 4)(2, 5, 6) = (1, 3, 5)(2, 4, 6),$$

then $(1, 3, 4)(2, 5, 6) = (1, 3, 5)(2, 4, 6) = (1, 5, 6)(2, 3, 4),$

by II., $(1, 3, 2)(4, 5, 6) = (1, 3, 6)(2, 4, 5) = (1, 4, 2)(3, 5, 6),$

and by IX.,

$$(1, 2, 6)(3, 4, 5) = (1, 5, 4)(2, 3, 6) = (1, 2, 5)(6, 3, 4),$$

and the lines 15, 36, and 24 are concurrent, and also 13, 26, and 45 are concurrent.

Similarly, the typical relation when $n = 4$—in other words, among the products of complementary tetrahedra which have their vertices at any eight points in the same tridimensional space (*i.e.*, in the same linear locus in space of four dimensions)—is

$$\begin{aligned}&(1, 2, 3, 4)(5, 6, 7, 8) + (1, 2, 3, 5)(6, 7, 8, 4)\\ &+ (1, 2, 3, 6)(7, 8, 4, 5) + (1, 2, 3, 7)(8, 4, 5, 6)\\ &+ (1, 2, 3, 8)(4, 5, 6, 7) = 0:\end{aligned}$$

these are ${}^8C_3$ or 56 in number, and should be dependent (Art. XX.) upon 23 independent relations.

By supposing two of the points to coincide, the following typical relation is obtained among the tetrahedra the vertices of which are situated at any seven points :

$$\begin{aligned}&(1, 2, 3, 4)(1, 5, 6, 7) + (1, 2, 3, 5)(6, 7, 4, 1)\\ &+ (1, 2, 3, 6)(7, 4, 1, 5) + (1, 2, 3, 7)(4, 1, 5, 6) = 0;\end{aligned}$$

and, by supposing two other points to coincide, this typical relation among the tetrahedra the vertices of which occupy any six points in space

$$\begin{aligned}(1, 2, 3, 4)(1, 2, 5, 6) &- (1, 2, 3, 5)(1, 2, 6, 4)\\ &+ (1, 2, 3, 6)(1, 2, 4, 5) = 0.\end{aligned}$$

In the geometry of space of n dimensions all the anharmonic systems up to the n^{th} have their application. Pencils of $2n$ straight lines through a point cutting a linear locus, form a system of the $\overline{n-1}^{\text{th}}$ class; pencils of $2n-2$ planes through a line cut linear loci represented by two equations in a system of the $\overline{n-2}^{\text{th}}$ class; and so on, until pencils of four linear loci through a common intersection cut a line in points which form an ordinary anharmonic system, *i.e.*, in accordance with the nomenclature above in one of the first class.

XLII. A special case, which seems to be the proper generalization of a harmonic pencil arises, when the pencil meeting at $(V, 0, 0, \ldots)$ is composed of the $2n$ straight lines

$$\nu = \pi = \ldots = 0, \quad \mu = \pi = \ldots = 0, \quad \mu = \nu = \ldots = 0, \text{ \&c.};$$

the edges of the simplicissimum of reference which meet at the origin of the pencil, and the other n lines

$$\frac{\mu}{-a} = \frac{\nu}{b} = \frac{\pi}{c} = \ldots, \quad \frac{\mu}{a} = \frac{\nu}{-b} = \frac{\pi}{c} = \ldots,$$

$$\frac{\mu}{a} = \frac{\nu}{b} = \frac{\pi}{-c} = \ldots, \text{ \&c.},$$

(or more generally of the $2n$ lines

$$N = 0, \ P = 0, \ldots; \quad M = 0, \ P = 0, \ldots; \quad M = 0, \ N = 0, \ldots; \text{ \&c.},$$

and

$$\begin{aligned} -AM &= BN = CP = \ldots, \\ AM &= -BN = CP = \ldots, \\ AM &= BN = -CP = \ldots, \\ &\text{\&c.} \qquad\quad \text{\&c.}, \end{aligned}$$

all of which meet at the point

$$M = 0, \ N = 0, \ P = 0, \ldots).$$

In space of three dimensions the anharmonic ratios corresponding to such a system are given by

$$(1,2,3)(4,5,6) : (1,2,4)(3,5,6) : (1,2,5)(3,4,6) : (1,2,6)(3,4,5)$$
$$: (1,3,4)(2,5,6) : (1,3,5)(2,4,6) : (1,3,6)(2,4,5) : (1,4,5)(2,3,6)$$
$$: (1,4,6)(2,3,5) : (1,5,6)(2,3,4) :: 2 : 1 : -1 : 0 : -1$$
$$: 0 : -1 : 1 : -1 : 0;$$

and the other systems obtained by changing the signs of a, b, or c will, like the above, have their anharmonic ratios

$$\pm 2, \ \pm\tfrac{1}{2}, \ \pm 1, \ 0, \text{ or } \infty.$$

The complete system, which includes all the rays of these, is composed of the first three of these rays, and four others, such that, if they cut any plane in the points 1, 2, 3 and 4, 5, 6, 7, respectively, then, if any three of the last four points be taken as the vertices of a triangle, the points 1, 2, 3 are the feet of concurrent lines from the vertices through the remaining point.

It is easily proved that, if the lines 56, 64, 45 touch a conic about 123 at these points respectively, the six points form a system of the kind, the seventh being the point of concurrence of 14, 25, 36. Therefore, if from any point three tangent planes be drawn to a quadric surface, the three points of contact, and the three points where the intersections of the tangent planes meet the polar plane of the original point, form a system of the kind stated.

Again, if two sets of points 4, 5, 6 and 4′, 5′, 6′ are each thus conjugate to the same three primary points 1, 2, 3, and if 7 and 7′ be the additional points connected with these systems, the eight points 4, 5, 6, 7 and 4′, 5′, 6′, 7′ all lie on the same conic, to which 123 is self-conjugate—viz., if the rays through 1, 2, 3 be the edges of the simplicissimum as above, and those through 4, 5, 6 the remaining three lines, these and that through 7, $\frac{\mu}{a} = \frac{\nu}{b} = \frac{\pi}{c}$ and 4′, 5′, 6′ and 7′ the same with a', b', c' written for a, b, c, on

$$\begin{vmatrix} \lambda^2, & \mu^2, & \nu^2 \\ a^2, & b^2, & c^2 \\ a'^2, & b'^2, & c'^2 \end{vmatrix} = 0.$$

In the same way, in space of n dimensions a pencil of $2n$ rays of the kind defined at the beginning of this article cuts any transversal linear locus in $2n$ points, such that first n points are the feet of concurrent lines from the vertices of the simplicissimum defined by the second n, and the anharmonic ratios are

$$\pm 2, \quad \pm \tfrac{1}{2}, \quad \pm 1, \quad 0, \quad \infty .$$

Any set of n points so conjugate with a given set of n points, involves, by changing the signs of a, &c., a system of 2^{n-1} points constituting 2^{n-1} sets of ratios.

All the $n \,.\, 2^{n-1}$ points belonging to n such systems conjugate to the same n points, lie upon the same quadric locus in space of $(n-1)$ dimensions, to which the simplicissimum, of which the vertices are at the n points to which each system is conjugate, is self-reciprocal; and if $a, b, c \ldots$; $a', b' \ldots$, &c. be the constants determining these points, this locus is

$$\begin{vmatrix} \lambda^2, & \mu^2, & \nu^2 & \ldots \\ a^2, & b^2, & c^2 & \ldots \\ a'^2, & b'^2, & c'^2 & \ldots \\ a''^2, & b''^2, & c''^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

Each system of points upon such a quadric is analogous to a system of points in involution: the vertices of the simplicissimum corresponding to the foci of the involution. Each point on the quadric determines the other 2^{n-1} points of its system.

It may also be shown that the four systems of this kind, determined on the faces of the tetrahedron of reference by the same point of concurrence, lie on the same quadric surface; for, if this point be $a : b : c : d$, the points on $\lambda = 0$ are

$$0 : -b : c : d, \quad 0 : b : -c : d, \quad 0 : b : c : -d;$$

those on $\mu = 0$,

$$-a : 0 : c : d, \quad a : 0 : -c : d, \quad a : 0 : c : -d$$

those on $\nu = 0$,

$$-a : b : 0 : d, \quad a : -b : 0 : d, \quad a : b : 0 : -d;$$

and on $\pi = 0$,

$$-a : b : c : 0, \quad a : -b : c : 0, \quad a : b : -c : 0$$

these all lie upon

$$\frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2} + \frac{\pi^2}{d^2} + 3\left\{\frac{\lambda\mu}{ab} + \frac{\lambda\nu}{ac} + \frac{\lambda\pi}{ad} + \frac{\mu\nu}{bc} + \frac{\mu\pi}{bd} + \frac{\nu\pi}{cd}\right\} = 0,$$

or
$$\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c} + \frac{\pi}{d}\right)^2 + \left(\frac{\lambda\mu}{ab} + \frac{\lambda\nu}{ac} + \frac{\lambda\pi}{ad} + \ldots\right) = 0,$$

which has double contact with the circumscribed quadric

$$\frac{\lambda\mu}{ab} + \frac{\lambda\nu}{ac} + \frac{\lambda\pi}{ad} + \frac{\mu\nu}{bc} + \frac{\mu\pi}{bd} + \frac{\nu\pi}{cd} = 0,$$

which passes through the points

$$(-a, b, c, d); \quad (a, -b, c, d); \quad (a, b, -c, d); \quad (a, b, c, -d).$$

In the same way, in space of four dimensions, the five systems determined on the faces of the simplicissimum of reference by the same point of concurrence,

$$(a : b : c : d : e),$$

all lie upon

$$\frac{\lambda\mu}{ab} + \frac{\lambda\nu}{ac} + \frac{\lambda\pi}{ad} + \frac{\lambda\rho}{ae} + \frac{\mu\nu}{bc} + \frac{\mu\pi}{bd} + \frac{\mu\rho}{be} + \frac{\nu\pi}{cd} + \frac{\nu\rho}{ce} + \frac{\pi\rho}{de} = 0;$$

and, in space of five dimensions, the systems determined by the point

$$(a : b : c : d : e : f)$$

lie upon $$\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}+\frac{\pi^2}{d^2}+\frac{\rho^2}{e^2}+\frac{\sigma^2}{f^2}$$

$$-\frac{5}{2}\left\{\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\frac{\lambda\pi}{ad}+\frac{\lambda\rho}{ae}+\frac{\lambda\sigma}{af}+\frac{\mu\nu}{bc}+\&c.\right\}=0;$$

and so on, the quadric locus being of the form

$$\left(\frac{\lambda}{a}+\frac{\mu}{b}+\ldots\right)^2-\kappa\left(\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}+\ldots\right)=0,$$

and having contact with

$$\left(\frac{\lambda\mu}{ab}+\frac{\lambda\nu}{ac}\ldots\ldots\right)=0$$

along its intersection with

$$\frac{\lambda}{a}+\frac{\mu}{b}+\ldots=0,$$

a locus of which interpretations are given in Art. XXIII. *ad finem.*

XLIII. Two quadric loci, which intersect in the linear locus at infinity and another linear locus, are similar and similarly situated, having two centres of similitude in the line of centres.

$$U=A_{11}\lambda^2+A_{22}\mu^2+A_{33}\nu^2+\ldots+2A_{12}\lambda\mu+\ldots=0\ldots\ldots\ldots(1)$$

and $$U+(B_1\lambda+B_2\mu+B_3\nu+\ldots)(\lambda+\mu+\nu+\ldots)=0\ \ldots\ldots\ldots(2)$$

are two such loci.

If $$\frac{\lambda-\lambda'}{a}=\frac{\mu-\mu'}{b}=\frac{\nu-\nu'}{c}=\ldots=\frac{Vp}{\sqrt{-S(a,b,c,\ldots)}}\equiv mp,$$

where $$(a+b+c+\ldots=0)\ \text{(Art. XVIII.)}$$

be any line through $(\lambda',\mu',\nu',\ldots)$ cutting (1) and (2) at distances p and p', these are determined by the equations

$$U'+mp\left(a\frac{dU'}{d\lambda'}+b\frac{dU'}{d\mu'}+c\frac{dU'}{d\nu'}+\ldots\right)$$

$$+m^2p^2(A_{11}a^2+A_{22}b^2+A_{33}c^2+\ldots+2A_{12}bc+\ldots)=0,$$

also, $$q(\lambda'+\mu'+\nu'+\ldots) = qV,$$

so that there are $n+3$ equations from which to eliminate

$$q\lambda',\ q\mu',\ q\nu' \ldots \text{ and } m,$$

and all but one are linear. Hence the resultant is a quadratic in q, each solution of which determines one position of $(\lambda', \mu', \nu', \ldots)$.

If $$(\lambda'', \mu'', \nu'', \ldots) \text{ and } (\lambda''', \mu''', \nu''', \ldots)$$

be the centres of (1) and (2),

therefore $$\frac{dU''}{d\lambda''} = \frac{dU''}{d\mu''} = \frac{dU'}{d\nu''} = \ldots,$$

and $$\frac{dU'''}{d\lambda'''} - B_1 V = \frac{dU'''}{d\mu'''} - B_2 V = \frac{dU'''}{d\nu'''} - B_3 V = \ldots;$$

and the line joining these points is therefore

$$\frac{\frac{dU}{d\mu} - \frac{dU}{d\lambda}}{B_2 - B_1} = \frac{\frac{dU}{d\nu} - \frac{dU}{d\lambda}}{B_3 - B_1} = \ldots,$$

and by (4) the centres of similitude lie upon this.

Other properties of similar and similarly situated quadric loci are given in Art. XXV. (note), Art. XXXI., and in the Addendum to Art. XIII., prefixed to this paper.

XLIV. Two quadric loci in space of n dimensions have a common self-conjugate simplicissimum. Let

$$U \equiv A_{11}\lambda^2 + A_{22}\mu^2 + A_{33}\nu^2 + \ldots + 2A_{12}\lambda\mu + \ldots = 0$$

$$W \equiv B_{11}\lambda^2 + B_{22}\mu^2 + B_{33}\nu^2 + \ldots + 2B_{12}\lambda\mu + \ldots = 0$$

be the quadrics, and let $(\lambda', \mu', \nu' \ldots)$ be a point which has the same linear polar with respect to each of them.

Therefore $$\lambda\frac{dU'}{d\lambda'} + \mu\frac{dU'}{d\mu'} + \nu\frac{dU'}{d\nu'} + \ldots = 0$$

and $$\lambda\frac{dW'}{d\lambda'} + \mu\frac{dW'}{d\mu'} + \nu\frac{dW'}{d\nu'} + \ldots = 0$$

are identical loci.

Therefore $$\frac{dU'}{d\lambda'} : \frac{dW'}{d\lambda'} :: \frac{dU'}{d\mu'} : \frac{dW'}{d\mu'} :: \frac{dU'}{d\nu'} : \frac{dW'}{d\nu'} :: \&c.,$$

$$:: r : 1, \text{ say};$$

and, eliminating λ', μ', ν', &c. from the resulting $n-1$ homogeneous linear equations

$$\left.\begin{array}{lllll} A_1-B_1r, & A_2-B_2r, & A_3-B_3r, & \ldots & \\ A_2-B_2r, & A_3-B_3r, & A_4-B_4r, & \ldots & \\ A_3-B_3r, & A_4-B_4r, & A_5-B_5r, & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \end{array}\right\} = 0 \quad \ldots\ldots\ldots\ldots (1)$$

an equation of the $(n-1)^{th}$ degree in r.

(It is, indeed, the discriminant of $U-rW=0$, and the values of r are those for which this quadric has a double point.)

Each root r_1, r_2 &c. of (1) gives a separate solution for $\lambda', \mu', \nu', \ldots$, so that, when the roots are all unequal, the $n+1$ corresponding points

$$(\lambda'_1, \mu'_1, \nu'_1 \ldots), \ (\lambda'_2, \mu'_2, \nu'_2 \ldots), \ (\lambda'_{n+1}, \mu'_{n+1}, \nu'_{n+1})$$

are the vertices of a simplicissimum, and each lies upon the linear polar of all the others with respect to both the quadrics.

For, if r_s and r_t be two values of r,

$$\frac{dU'_s}{d\lambda'_s} = r_s \frac{dW'_s}{d\lambda'_s}, \quad \frac{dU'_s}{d\mu'_s} = r_s \frac{dW'_s}{d\mu'_s}, \text{ &c.},$$

and

$$\frac{dU'_t}{d\lambda'_t} = r_t \frac{dW'_t}{d\lambda'_t}, \quad \frac{dU'_t}{d\mu'_t} = r_t \frac{dW'_t}{d\mu'_t}, \text{ &c.};$$

therefore

$$\lambda'_t \frac{dU'_s}{d\lambda'_s} + \mu'_t \frac{dU'_s}{d\mu'_s} + \nu'_t \frac{dU'_s}{d\nu'_s} + \ldots$$

$$= r_s \left\{ \lambda'_t \frac{dW'_s}{d\lambda'_s} + \mu'_t \frac{dW'_s}{d\mu'_s} + \nu'_t \frac{dW'_s}{d\nu'_s} + \ldots \right\},$$

or

$$\lambda'_s \frac{dU'_t}{d\lambda'_t} + \mu'_s \frac{dU'_t}{d\mu'_t} + \nu'_s \frac{dU'_t}{d\nu'_t} + \ldots$$

$$= r_s \left\{ \lambda'_s \frac{dW'_t}{d\lambda'_t} + \mu'_s \frac{dW'_t}{d\mu'_t} + \nu'_s \frac{dW'_t}{d\nu'_t} + \ldots \right\};$$

but

$$\lambda'_s \frac{dU'_t}{d\lambda'_t} + \mu'_s \frac{dU'_t}{d\mu'_t} + \nu'_s \frac{dU'_t}{d\nu'_t} + \ldots$$

$$= r_t \left\{ \lambda'_s \frac{dW'_t}{d\lambda'_t} + \mu'_s \frac{dW'_t}{d\mu'_t} + \nu'_s \frac{dW'_t}{d\nu'_t} + \ldots \right\},$$

and therefore, r_s and r_t being different,

$$\lambda'_s \frac{dU'_t}{d\lambda'_t} + \mu'_s \frac{dU'_t}{d\mu'_t} + \nu'_s \frac{dU'_t}{d\nu'_t} + \ldots = 0,$$

$$\lambda'_s \frac{dW'_t}{d\lambda'_t} + \mu'_s \frac{dW'_t}{d\mu'_t} + \nu'_s \frac{dW'_t}{d\nu'_t} + \ldots = 0,$$

and $(\lambda_i', \mu_i', \nu_i' \ldots)$ lies upon the polar of $(\lambda_i', \mu_i', \nu_i' \ldots)$ with respect to each of the quadrics $U = 0$ and $W = 0$.

Hence, when the roots of (1) are unequal, these two quadrics have a definite self-conjugate simplicissimum, the vertices of which are the double points of those quadrics of the system $U - rW = 0$ which have double points.

When two or more of the roots of (1) are equal, the corresponding vertices are no longer determinate. If two be equal, one of the quadrics $U - rW = 0$ is expressible as a linear homogeneous function of $n-1$ squares, and has double points along a straight line; if three of the roots be equal, one of the quadrics is expressible as a linear homogeneous function of $n-2$ squares, and has double points over a plane; and so on.

For if, when referred to a common self-conjugate simplicissimum,

$$U \equiv a\lambda^2 + b\mu^2 + c\nu^2 + d\pi^2 + e\rho^2 + \ldots = 0,$$

$$W \equiv a'\lambda^2 + b'\mu^2 + c'\nu^2 + d'\pi^2 + e'\rho^2 + \ldots = 0,$$

it is necessary and sufficient, in order that

$$a\lambda'\lambda + b\mu'\mu + c\nu'\nu + d\pi'\pi + e\rho'\rho + \ldots = 0$$

and

$$a'\lambda'\lambda + b'\mu'\mu + c'\nu'\nu + d'\pi'\pi + e'\rho'\rho + \ldots = 0$$

may be identical, that all but one of the quantities $\lambda', \mu', \nu', \ldots$ should vanish, unless some of the coefficients be such that some of the ratios $a : b : \ldots :: a' : b' : c' \ldots$ hold—*i.e.*, unless the equation (1) has equal roots. When this is the case, it is only necessary that those coordinates, of which the coefficients do not satisfy the proportions above, should vanish. Thus, if $a : b :: a' : b'$, all the coordinates but λ' and μ' must vanish; if $a : b : c :: a' : b' : c'$, all but λ', μ', ν'; and so on.

If, in the first case, $(\lambda_1, \mu_1, 0, 0)$, $(\lambda_2, \mu_2, 0, 0)$ be new positions of the two vertices on (1.2),

$$a\lambda_1\lambda_2 + b\mu_1\mu_2 = 0,$$

and if $(\lambda_1, \mu_1, \nu_1, 0)$, $(\lambda_2, \mu_2, \nu_2, 0)$, $(\lambda_3, \mu_3, \nu_3, 0)$ be new positions of the three vertices in the plane (1, 2, 3),

$$a\lambda_1\lambda_2 + b\mu_1\mu_2 + c\nu_1\nu_2 = 0,$$

$$a\lambda_2\lambda_3 + b\mu_2\mu_3 + c\nu_2\nu_3 = 0,$$

$$a\lambda_3\lambda_1 + b\mu_3\mu_1 + c\nu_3\nu_1 = 0;$$

and similar relations hold in other cases.

The intersections of quadric loci in space of n dimensions may be

classified according as the roots (1) are equal or unequal. Thus, in ordinary space,

(i.) all the roots may be unequal,

(ii.) two roots may be equal and $W = AU + LM$,

(iii.) three roots may be equal and $W = AU + L^2$,

(iv.) two pairs of roots may be equal and

$$U = ALM + BL'M',$$

$$W = A'LM + BL'M';$$

and so for space of any dimensions.

XLV. If the transformation of Art. V. be applied to $U = 0$, the result, the equation in simplicissimum content coordinates to the intersection of $U = 0$ with the linear locus determined by the $p+1$ points

$$(x_1, y_1, z_1, \ldots), \quad (x_2, y_2, z_2, \ldots), \ \ldots \ (x_{p+1}, y_{p+1}, z_{p+1}),$$

will be

$$U_{11}\lambda^2 + U_{22}\mu^2 + U_{33}\nu^2 + \ldots + 2U_{12}\lambda\mu + \ldots = 0 \ \ldots\ldots\ldots\ldots \text{(v.)},$$

where U_{11} stands for the value of U when $x_1, y_1, z_1, \ldots$ are written for $x, y, z, \ldots$ and $2U_{12}$ for the result of operating on this with

$$\left(x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1} + \ldots\right).$$

If $p+1 = n$, so that the locus determined by the points is a linear locus in space of n dimensions, and then, if this be the tangent locus at $(x_1, y_1, z_1, \ldots)$, a point on $U = 0$,

$$U_{11} = 0, \quad U_{12} = 0, \quad U_{13} = 0 \ \ldots,$$

and the discriminant of (1) vanishes; therefore the linear tangent locus at any point on a quadric locus, in space of n dimensions, intersects the quadric in a quadric in space of $n-1$ dimensions (*i.e.*, on the linear locus), having a double point at the point of contact (comp. Art. XXXIII.); and if the locus through the n points be a tangent, the discriminant of (1) vanishes; and conversely, for if the discriminant of (1) vanishes, (1) has a node, and if this be taken as the first vertex of the simplicissimum of n vertices

$$U_{11} = U_{12} = U_{13} = \ldots = 0,$$

and the linear locus touches at this point.

Since the equation to the linear locus is

$$\begin{vmatrix} x, & y, & z, & \dots \\ x_1, & y_1, & z_1, & \dots \\ x_2, & y_2, & z_2, & \dots \\ \dots & \dots & \dots & \\ x_n, & y_n, & z_n, & \dots \end{vmatrix} \equiv ax+\beta y+\gamma z+\dots = 0 \quad \dots\dots\dots\dots \quad (2),$$

it follows that the vanishing of the discriminant of (1) is the condition that (2) should touch $U=0$, and therefore α, β, γ, &c., the first minors of the determinant in (2), must satisfy the reciprocal equation $\sigma = 0$; and hence the discriminant of (1) can only differ by a simple multiplier from the value of σ when the above values are written for α, β, γ, &c. (*Educational Times*, Questions 8940, 8970 and 9828).

By an entirely similar proof it appears that, if

$$\Sigma \equiv \sigma_{11}\alpha^2+\sigma_{22}\beta^2+\sigma_{33}\gamma^2+\dots+2\sigma_{23}\alpha\beta+\dots,$$

the discriminant of Σ can only differ by a multiplier from $\Delta^{n-1}U$, with the first minors of

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \dots \\ \alpha_1, & \beta_1, & \gamma_1, & \dots \\ \alpha_2, & \beta_2, & \gamma_2, & \dots \\ \dots & \dots & \dots & \\ \alpha_n, & \beta_n, & \gamma_n, & \dots \end{vmatrix}$$

written for x, y, z, &c.

If $$p = 1, \quad U_{11}\lambda^2+2U_{12}\lambda\mu+U_{22}\mu^2 = 0$$

determines the points in which the line joining the two given points cuts $U=0$; and, if $U_{11}U_{22}-U_{12}^2 = 0$, this line is a tangent, and therefore

$$4U_{11}U-\left\{\left(x\frac{d}{dx_1}+y\frac{d}{dy_1}+z\frac{d}{dz_1}+\dots\right)U_{11}\right\}^2 = 0$$

is the equation to the group of linear tangents from the point $(x_1, y_1, z_1, \dots)$.

If $p = 2$, the plane through these given points cuts $U=0$ in

$$U_{11}\lambda^2+U_{22}\mu^2+U_{33}\nu^2+2U_{12}\lambda\mu+\dots = 0,$$

and this will touch if

$$U_{11}U_{22}U_{33}+2U_{12}U_{13}U_{23}-U_{11}U_{23}^2-U_{22}U_{13}^2-U_{33}U_{12}^2 = 0;$$

and

$$\left(x_1\frac{\partial}{\partial x}+x_2\frac{\partial}{\partial y}+x_3\frac{\partial}{\partial z}+\dots+x_n\frac{\partial}{\partial w}\right)U$$

$$\left(x_1\frac{\partial}{\partial x}+x_2\frac{\partial}{\partial y}+x_3\frac{\partial}{\partial z}+\dots+x_n\frac{\partial}{\partial w}\right)V$$

$$\left(x_1\frac{\partial}{\partial x}+x_2\frac{\partial}{\partial y}+x_3\frac{\partial}{\partial z}+\dots+x_n\frac{\partial}{\partial w}\right)W=0,$$

is the equation to the group of tangent planes which can be drawn to $U=0$ from the line joining $(x_1, y_1, z_1, \dots)$ and $(x_2, y_2, z_2, \dots)$; and so on.

XLVI. *The $(n-1)^{\text{ary}}$ quantic which is the cissoid of the equation to a quadric cone in space of n dimensions has only one invariant, the discriminant: but two such quantics have in all $(n-2)$ invariants—the two discriminants and $n-3$ other functions involving the coefficients of both—these last are the coefficients of the powers $1, 2, \dots n$ of r in (1), Art. XLIV. They are combinants and are derived from the discriminant of U by operating upon that function with*

$$\frac{1}{p!}\left(B_{11}\frac{d}{dA_{11}}+B_{22}\frac{d}{dA_{22}}+\dots+B_{12}\frac{d}{dA_{12}}+\dots\right),$$

giving to p all values from 1 to n.

As stated above, the roots of (1) give the quadrics through the intersection of $U=0$ and $W=0$, which have double points.

If $a_{11}, a_{12}, \dots a_{1n}$, &c. be the first minors of the discriminant of U, the coefficient of r is

$$B_{11}a_{11}+B_{22}a_{22}+\dots+2B_{12}a_{12}+\dots,$$

and this vanishes if a simplicissimum self-conjugate with respect to $W=0$ can be inscribed in $U=0$; for then

$$B_{12}=B_{13}=B_{23}=\dots=0 \quad \text{and} \quad a_{11}=a_{22}=\dots=0 \text{ (Art. XXX.).}$$

If $V=0$ and $X=0$ be the equations to the sections of $U=0$ and $W=0$ respectively, by the linear locus

$$\alpha\lambda+\beta\mu+\gamma\nu+\dots+\theta\tau=0,$$

i.e., through the n points

$$\left(\frac{V\theta}{\theta-\alpha}, 0, 0, \dots \frac{-V\alpha}{\theta-\alpha}\right), \quad \left(0, \frac{V\theta}{\theta-\beta}, 0, \dots \frac{-V\beta}{\theta-\beta}\right), \text{ &c. (Art. V.),}$$

—the discriminant of $V-rX=0$ will be an equation in r of order n, the first and last coefficients when reduced being the discriminants of V and X; and so the sinisters of the reciprocal equations to $U=0$ and $W=0$, the meaning of the others may be obtained from the geometry of $(n-1)$-dimensions. Thus the vanishing of the coefficient of r implies, from the general result above, that all linear loci, the coefficients of which make it vanish, cut $U=0$, $W=0$ in sections such that a simplicissimum of n vertices self-conjugate with respect to the section of $W=0$ is inscribed in that of $U=0$.

In this way I interpreted the contravariants of two quadric surfaces in a paper read here in December 1883; and so, if U and W be quinary quadrics, and

$$\sigma-r\tau+r^2\pi-r^3\tau'+r^4\sigma'=0$$

(where σ and σ' are the reciprocals of U and W), the expanded form of the discriminant of $V-rX=0$, σ, τ, π, τ', σ' are the Δ, θ, ϕ, θ', Δ' of the quadric surfaces $V=0$ and $X=0$ (Salmon, *Geometry of Three Dimensions*, p. 145).

If $\tau=0$, as above, it is possible to inscribe in $V=0$ a tetrahedron self-conjugate with respect to $X=0$.

If $\tau'=0$, there is a tetrahedron self-conjugate with respect to $V=0$, the faces of which touch $X=0$.

If $\pi=0$, the edges of a tetrahedron, self-conjugate with respect to either $V=0$ or $X=0$, touch the other.

The condition that a tetrahedron may be inscribable in $X=0$, which has two pairs of opposite edges on $V=0$, is

$$4\sigma\tau\pi=\tau^3+8\sigma^2\tau',$$

and that it may be possible to find a tetrahedron having two pairs of opposite edges on one of the quadrics and its four faces touching the other

$$4\sigma'\tau'\pi=\tau'^3+8\sigma'^2\tau.$$

It thus appears that two $(n+1)$-ary quadrics have $(n+2)$ independent invariants, and $(n+1)$ independent contravariants.

If $u=0$ and $w=0$ be the tangential equations to $U=0$ and $W=0$ respectively, $u-rw=0$ will be the tangential equation to the system of quadrics which touch the same system of common tangent loci as $U=0$ and $W=0$.

The equation to $u-rw=0$ in simplicissimum coordinates will be

[illegible]

I regret to have to add the following corrections to the present paper:—

p. 472, l. 3, *for* $\frac{1-\frac{1}{p}-\frac{1}{q}}{p-1}\Sigma\frac{1}{\alpha}$ *read* $\frac{1-\frac{1}{p}-\frac{1}{q}}{p-1}\Sigma\frac{1}{A_r}$.

p. 473, l. 11, *for* σ, *read* σ^2.

p. 473, l. 13, *for* $\frac{1}{\sigma^2}$ *read* $\frac{k}{\sigma^2}$.

l. 15, *for* kp^2, *read* $kp^2\sigma^2$.

p. 474, l. 14, *for* CC' *read* $4CC'$.

l. 17, *for* CC, *read* $4CC$.

p. 474, l. 27, *for* Q_1, *read* Q_1^2.

[1911. It is only during the printing of this paper that I have noticed Mr. Brill's question (8431) and its solution in the *Reprint* from the *Educational Times*, Vol. XLVIII., p. 111. In this the equation of [illegible] are enunciated for six points on a circle.]

On Modular Equations. *By* Mr. R. Russell.

[*Read June 12th*, 1890.]

The theory of modular equations has since the time of Jacobi and Legendre excited a great deal of interest among mathematicians. The form in which Jacobi expressed them, viz., in terms of $u = \sqrt[4]{\kappa}$ and $v = \sqrt[4]{\lambda}$ was found to lead to inconveniently long expressions, and has to a great extent been abandoned; and forms suggested by Legendre's equation of the third order

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} - 1 = 0,$$

and Gutzlaff's equation of 7th order

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} - 1 = 0,$$

have in most cases taken their place.

I had known the results for $n = 3, 5, 7, 11, 23, 31$, and in endeavouring to obtain proofs of them was led to write the paper which appeared in Vol. XIX., *Proceedings* of the Society, where will be found the modular equations for all the prime orders of transformation up to $n = 47$, with the exceptions of 29, 41, 43.

In this article the same notation will be used, but in most cases a different method of forming the modular equations will be adopted, which will enable us to obtain the results for $n = 13, 17, 29, 41, 43, 47, 53, 59, 71$, and others, with comparatively little labour. This method, involving as it does the knowledge of the equations for $2\kappa\kappa'$, $2\sqrt{\kappa\kappa'}$, $2\sqrt[4]{\kappa\kappa'}$, when $\omega \equiv \frac{iK'}{K} = \sqrt{-n}$, introduces the problem of *complex multiplication.* The modular equations in a few of the well-known cases will be presented in new forms more suitable for the purposes of complex multiplication than those usually employed, and by their assistance I shall calculate the equations for determining the moduli in a few new and some well-known cases.

1. Transformation of the second order:—The transformation

$$y = \frac{(1+\kappa)x}{1+\kappa x^2}$$

leads at once to the well-known relations

$$\lambda = \frac{2\sqrt{\kappa}}{1+\kappa}, \quad \lambda' = \frac{1-\kappa}{1+\kappa}, \quad \text{where} \quad \sqrt[4]{\lambda} = \phi\left(\frac{\omega}{2}\right), \text{ and } \sqrt[4]{\kappa} = \phi(\omega),$$

or, as it is sometimes written,

$$\sqrt{\lambda\kappa'} = \sqrt{2}\sqrt[4]{\lambda\kappa'}.$$

When expressed in terms of Hermite's functions these become

$$\phi^4\left(\frac{\omega}{2}\right) = \frac{2\phi^2(\omega)}{1+\phi^4(\omega)}, \quad \psi^4\left(\frac{\omega}{2}\right) = \frac{1-\phi^4(\omega)}{1+\phi^4(\omega)} = \left[\frac{1-\phi^4(\omega)}{\psi^2(\omega)}\right]^2,$$

$$\phi^4(2\omega) = \frac{1-\psi^4(\omega)}{1+\psi^4(\omega)} = \left[\frac{1-\psi^4(\omega)}{\phi^2(\omega)}\right]^2, \quad \psi^4(2\omega) = \frac{2\psi^2(\omega)}{1+\psi^4(\omega)},$$

$$\phi^2(\omega)\,\psi^2(2\omega) = \sqrt{2}\,\phi(2\omega)\,\psi(\omega);$$

whence it at once follows that

$$\phi^2(4\omega) = \frac{1-\psi^2(\omega)}{1+\psi^2(\omega)}.$$

2. If $$\sqrt[4]{\lambda} = \phi(\Omega), \quad \sqrt[4]{\kappa} = \phi(\omega);$$

and if $$\omega = 4s \pm 1 - \frac{2}{\Omega+1},$$

it is very easy to prove that

$$\sqrt[4]{4\kappa\lambda\kappa'\lambda'} = e^{\frac{1}{2}(2s\omega)} \text{ or } e^{(2s-1)\frac{1}{2}i\omega},*$$

according as we take the upper or lower sign.

From the relation $$\omega = 4s - 1 - \frac{2}{\Omega+1},$$

with its equivalent $$\sqrt[4]{4\kappa\lambda\kappa'\lambda'} = e^{(2s-1)\frac{1}{2}i\omega},$$

can be deduced a very important formula from which flow many results in the theory of complex multiplication. It really consists in obtaining a relation between $2\kappa\kappa'$ and $2\kappa\lambda$.

We have $$\phi(\omega) = e^{(4s-1)\frac{1}{4}i\omega}\frac{\psi\left(\frac{\Omega+1}{2}\right)}{\phi\left(\frac{\Omega+1}{2}\right)};$$

therefore $$\kappa = e^{-\frac{1}{2}i\omega}\frac{1-\phi^4(\Omega+1)}{2\phi^2(\Omega+1)} = e^{-\frac{1}{2}i\omega}\frac{\lambda'-i\lambda}{2\sqrt{\lambda\lambda'}};$$

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* Is an immediate deduction from the above, combined with the elementary relations connecting Hermite's functions, viz.,

$$\phi(\omega+1) = e^{\frac{1}{4}i\omega}\frac{\phi(\omega)}{\psi(\omega)}, \quad \psi(\omega+1) = \frac{1}{\psi(\omega)},$$

$$\phi\left(-\frac{1}{\omega}\right) = \psi\omega, \quad \psi\left(-\frac{1}{\omega}\right) = \phi(\omega).$$

hence $$2\kappa\lambda e^{\frac{1}{4}\omega} = (\lambda' - i\lambda)\sqrt{\frac{\lambda}{\lambda'}},$$

and $$e^{\frac{1}{4}\omega}\left(2\kappa\lambda + \frac{1}{2\kappa\lambda}\right) = (\lambda' - i\lambda)\sqrt{\frac{\lambda}{\lambda'}} - i(\lambda' + i\lambda)\sqrt{\frac{\lambda'}{\lambda}}$$

$$= \frac{2\lambda\lambda' - \sqrt{-1}}{\sqrt{\lambda\lambda'}} = \left(\sqrt{2\lambda\lambda'} - \frac{\sqrt{-1}}{\sqrt{2\lambda\lambda'}}\right)\sqrt{2}:$$

but $$\sqrt{2\kappa\kappa'}\,\sqrt{2\lambda\lambda'} = e^{(2s-1)\frac{1}{4}\omega};$$

therefore $$e^{\frac{1}{4}\omega}\left(2\kappa\lambda + \frac{1}{2\kappa\lambda}\right) = e^{(2s-1)\frac{1}{4}\omega}\sqrt{2}\left(\frac{1}{\sqrt{2\kappa\kappa'}} + \sqrt{-1}\sqrt{2\kappa\kappa'}\right).$$

If, therefore, in the modular equation of the nth order we put

$$\sqrt[4]{4\kappa\lambda\kappa'\lambda'} = e^{(2s-1)\frac{1}{4}\omega},$$

we obtain an equation for $2\kappa\lambda$ which will be reciprocal, and from which can be deduced one for $\frac{1}{2\kappa\kappa'} \pm 2\kappa\kappa'$, corresponding to those values of ω given by the relation

$$\omega = 4s - 1 - \frac{2}{\frac{\omega + 16r}{n} + 1},$$

or $$\omega = -\left(8r - 2s + \frac{n+1}{2}\right) + \sqrt{-\left[2n - \left(8r + 2s + \frac{n-1}{2}\right)^2\right]}.$$

Prof. Greenhill, by applying this method to the case of $n = 19$, obtained the equation for $\frac{1}{2\kappa\kappa'} - 2\kappa\kappa'$, where $\omega = \sqrt{-29}$ (see *Proceedings*, Vol. XIX., p. 328). Had he taken the modular equation of the 15th order instead, the equation would have been more easily obtained, inasmuch as no extraneous factors occur.

Taking $n = 15$, $r = -1$, $s = 1$, we have

$$\omega = 2 + \sqrt{-29};$$

hence, if in the equation $(P^2 - 4Q)P + 4R = 0$

we put $$\sqrt[4]{2\kappa'\lambda'} = \frac{e^{\frac{1}{4}\omega}}{\sqrt[4]{2\kappa\lambda}},$$

we obtain for $$t \equiv \left(\sqrt[4]{2\kappa\lambda} + \frac{e^{\frac{1}{4}\omega}}{\sqrt[4]{2\kappa\lambda}}\right)$$

the equation $t^8-t^3\sqrt[4]{2}-(3+2\sqrt{-1})\sqrt{2}\,t+\sqrt[4]{8}=0$.

Rationalising by squaring twice, and then replacing $2\kappa\lambda+\frac{1}{2\kappa\lambda}$ by its value in terms of $\sqrt{2\kappa\kappa'}$, we have Greenhill's equation

$$\beta^3-588\beta^2-976\beta-3136=0, \quad \beta=\frac{1}{2\kappa\kappa'}-2\kappa\kappa'.$$

This method (practically the same as Hermite's) labours under the disadvantage that the simplest equation we can obtain by it is expressed in terms of $2\kappa\kappa'$, and consequently the coefficients will be very large, whereas in most cases the equation we wish to obtain is one either for $\frac{1}{\sqrt[3]{2\kappa\kappa'}}-\sqrt[3]{2\kappa\kappa'}$ or $\frac{1}{\sqrt[6]{2\kappa\kappa'}}+\sqrt[6]{2\kappa\kappa'}$. Under the head of complex multiplication, I shall give some theorems for calculating these equations in terms of the quantities just mentioned.

3. For convenience of reference, I give at once some formulæ which will be found useful in determining coefficients in the modular equations

$$\sqrt[4]{\kappa}=\sqrt{2}\,q^{\frac{1}{8}}(1-q+2q^2-3q^3+4q^4-6q^5+9q^6\ldots),$$

$$\frac{1}{\sqrt[4]{\kappa}}=\frac{1}{\sqrt{2}\,q^{\frac{1}{8}}}(1+q-q^2+q^4-q^6\ldots),$$

$$\frac{1}{\sqrt{\kappa}}=\frac{1}{2q^{\frac{1}{4}}}(1+2q-q^3-2q^3+3q^4+2q^5-4q^6\ldots),$$

$$\frac{1}{\kappa}=\frac{1}{4q^{\frac{1}{2}}}(1+4q+2q^2-8q^3-q^4+20q^5-2q^6\ldots),$$

$$\frac{1}{\kappa^2}=\frac{1}{16q}(1+8q+20q^2-62q^4+216q^6\ldots),$$

$$\sqrt[4]{\kappa'}=1-2q+2q^2-4q^3+6q^4-8q^5+12q^6\ldots$$

$$\sqrt{\kappa'}=1-4q+8q^2-16q^3+32q^4-56q^5+96q^6\ldots,$$

$$\kappa'=1-8q+32q^2-96q^3+256q^4-624q^5+1408q^6\ldots,$$

$$\sqrt[12]{2\kappa\kappa'}=\sqrt[4]{2}\,q^{\frac{1}{6}}(1-q+q^2-2q^3+2q^4-3q^5+4q^6\ldots),$$

$$\frac{1}{\sqrt[12]{2\kappa\kappa'}}=\frac{1}{\sqrt[4]{2}\,q^{\frac{1}{6}}}(1+q+q^2+q^4+q^5+q^6+\ldots).$$

Also, from the transformations of the second order we have

$$\frac{4\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2} \cdot \frac{\psi^2(2\omega)}{\varphi^2(2\omega)} = -\frac{1}{\varphi^8(2\omega+1)},$$

$$\sqrt[3]{\frac{1-\kappa'}{4\kappa\kappa'}} = \sqrt[3]{\frac{(1-\kappa')^2}{4\kappa'} \cdot \frac{1-\kappa'}{\kappa}} = \sqrt[3]{-\varphi^8(2\omega+1)\varphi^2(2\omega)} = 2\sqrt[3]{4}q^{\frac{1}{3}}f(q^3),$$

$$\frac{4\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2} = \frac{4\psi^2(\omega)}{[1-\psi^2(\omega)]^2} = \frac{1}{\varphi^4(4\omega)} - 1,$$

$$\frac{1-\sqrt{\kappa'}}{\sqrt{\kappa}} = \frac{1-\psi^2(\omega)}{\varphi^2(\omega)} = \varphi(4\omega)\,\varphi(2\omega).$$

These functions can therefore be expanded very quickly in powers of q, since they are all functions of 2ω or 4ω, that is, of q^2 or q^4. Their values are, as far as we shall require them,—

$$\frac{4\kappa'}{(1-\kappa')^2} = \frac{1}{16q^2}(1-8q^2+20q^4-62q^6+216q^{12}\ldots),$$

$$\sqrt[3]{\frac{1-\kappa'}{4\kappa\kappa'}} = 2\sqrt[3]{4}\,q^{\frac{1}{3}}(1+q^2+3q^4+4q^6+7q^8+10q^{10}+\ldots)^2,$$

$$\frac{4\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2} = \frac{1}{4q^2}(1-4q^2+4q^4+2q^8\ldots),$$

$$\frac{1-\sqrt{\kappa'}}{\sqrt{\kappa}} = 2q^{\frac{1}{2}}(1-q^2+q^4-2q^6+4q^8\ldots).$$

4. The modular equations for *prime* order of transformation may be divided into four classes, according as $\frac{n+1}{8}$, when reduced to its simplest value, assumes the form $2p$, p, $\frac{p}{2}$, or $\frac{p}{4}$, and are homogeneous symmetric functions

of $\sqrt[4]{\kappa\lambda}$, $\sqrt[4]{\kappa'\lambda'}$, $+1$, of degree $2p$ when $\frac{n+1}{8} = 2p$, (Class I.),

of $\sqrt[4]{\kappa\lambda}$, $\sqrt[4]{\kappa'\lambda'}$, -1, of degree p when $\frac{n+1}{8} = p$ (odd), (Class II.),

of $\sqrt{\kappa\lambda}$, $\sqrt{\kappa'\lambda'}$, -1, of degree p when $\frac{n+1}{8} = \frac{p}{2}$ (odd), (Class III.),

of $\kappa\lambda$, $\kappa'\lambda'$, -1, of degree p when $\frac{n+1}{8} = \frac{p}{4}$ (odd), (Class IV.).

(See *Proceedings*, Vol. XIX., pp. 93–95.)

The modular equation for $n = 29$ is therefore of the 15th degree

in $\kappa\lambda$, $\kappa'\lambda'$. -1, and contains 19 unknown coefficients. At very great labour I had determined all but one of these, but the calculations necessary to obtain the 19th coefficient were so repulsive that I decided to adopt another method, and thus recognised the following very important theorem :—

If $n+1$ be divisible by 3, then the corresponding modular equation always reduces to the form

$$U^3+V^3+W^3-3UVW=0,$$

and therefore to $$U+V+W=0.$$

The proof of the theorem is very easy—for the modular equation may really be considered as a relation between $x+y$ and xy, where x and y are $\sqrt[4]{\kappa\lambda}$, $\sqrt[4]{\kappa'\lambda'}$; $\sqrt{\kappa\lambda}$, $\sqrt{\kappa'\lambda'}$; $\kappa\lambda$, $\kappa'\lambda'$, according to the class to which it belongs. Now, since

$$\sqrt[12]{4\kappa\kappa'\lambda\lambda'}=\sqrt[4]{2}\,q^{\frac{1}{3}(n+1)}f(q),$$

we see that, when $n+1\equiv 0 \pmod 3$, the relation becomes one between $x+y$ and $\sqrt[3]{xy}$, which, when rationalised, must reduce to the old form,—in other words, the equation reduces to

$$U^3+V^3+W^3-3UVW=0,$$

or $$U+V+W=0.$$

5. *Modular Equations of the 17th order.*—The equation of the 17th order, which I had already calculated in the form

$$2^{21}\cdot 3^3R^3+2^{10}R^2(7309P^3-2^8\cdot 117PQ)$$
$$+2^6R(-287P^6+2^5\cdot 261P^4Q-2^{12}\cdot 15P^2Q^2+2^{17}Q^3)+P^9=0,$$

may therefore, in the light of the theorem just proved, be written

$$384R+52P(4R)^{\frac{1}{3}}+(2^7Q-20P^2)(4R)^{\frac{2}{3}}+P^3=0;$$

and when we put $\lambda=\kappa'$, $\lambda'=\kappa$, $z=(2\kappa\kappa')^{\frac{1}{4}}$, we get

$$(z-1)(z^3-3z-1)^2(z^4-5z^3+4z^2-5z+1)=0.$$

When $n+1\equiv 0 \pmod 3$, the modular equations are reducible to this simplified form, in whatever notation they may be expressed. I find, by direct use of the "q formulæ," that the equation of the 17th order may also be written

$$(\sqrt[4]{\lambda\kappa'}-\sqrt[4]{\lambda'\kappa})^6+10t^2(\sqrt[4]{\lambda\kappa'}-\sqrt[4]{\lambda'\kappa})^2+4\sqrt{2}\,t(t^4-1)=0,$$
$$(t=\sqrt[12]{4\kappa\kappa'\lambda\lambda'}),$$

from which we may at once deduce Sohnke's equation. For, if we assume $\omega = \sigma + 1$, then $\Omega = 17\sigma + 17$; therefore

$$\sqrt[4]{\lambda\kappa'} - \sqrt[4]{\lambda'\kappa} = e^{\frac{1}{24}\pi i} \frac{\phi(17\sigma) - \phi(\sigma)}{\psi(\sigma)\,\psi(17\sigma)} = e^{\frac{1}{24}\pi i} \frac{v-u}{v'u'},$$

$$\sqrt[12]{\lambda\kappa\lambda'\kappa'} = e^{\frac{1}{24}\pi i} \left(\frac{vu}{v'^2u'^2}\right)^{\frac{1}{2}};$$

and therefore the above equation assumes the form

$$(v-u)^6 - 2\sqrt[3]{2}\left[5(v-u)^2 + 4vu\right](vuv'u')^{\frac{1}{2}}(v'u')^2$$
$$-4\sqrt[3]{4}\,(vuv'u')^{\frac{1}{2}}(v'u')^5 = 0,$$

or
$$\frac{(v-u)^6}{(vu)^3} - \left(\frac{2v'^2u'^2}{vu}\right)^{\frac{1}{2}} \cdot \frac{5(v-u)^2 + 4vu}{vu} - \left(\frac{2v'^2u'^2}{vu}\right)^2 = 0.$$

This is the result we should get if Sohnke's equation were reduced to the form

$$U^3 + V^3 + W^3 - 3UVW = 0,$$

and the factor $U + V + W$ abstracted; or, in other words, if we rationalise the equation just obtained, we shall have Sohnke's result.

6. *Modular Equation of the 71st order.*—Similarly, Fiedler's equation of the 71st order,

$$P^6 - 4R\left[(P^2-4Q)^3 + 9P^2(P^2-4Q)^2 + 21P^4(P^2-4Q) + 12P^6\right]$$
$$-16R^2P\left[6(P^2-4Q) + 7P^2\right] - 64R^3 = 0,$$

was obtained by me in the form

$$-4R + 2(4R)^{\frac{1}{2}}P + 4(4R)^{\frac{1}{2}}(Q-P^2) + P^3 = 0,*$$

and may be verified very easily.

The modular equations for $n = 41, 47, 53, 59, 83, 89, 113, 131, 137, 143$, &c. ... follow the same law.

Complex Multiplication Moduli.

7. I. If in any modular equation included in Classes I., II., we put

$$\sqrt[4]{\lambda} = \sqrt[4]{\kappa'} \text{ and } \sqrt[4]{\lambda'} = \sqrt[4]{\kappa},$$

we obtain an equation for

$$\sqrt[4]{16\lambda\kappa'} \text{ or } \sqrt[12]{16\kappa\kappa'}$$

corresponding to those values of ω given by the relation

$$\omega = 8p + \sqrt{-(n-64p^2)};$$

* I have since discovered that Weber gives the same form for $n = 71$ (see *Acta Mathematica*, Vol. XI.).

[illegible] the equation $\zeta' = \zeta^n$ [illegible]

$$\frac{\omega - 2p}{\kappa} = -\frac{1}{\omega} - 2s,$$

therefore

$$\frac{\omega - 2p}{\kappa} = -\frac{1}{\omega} - 2s,$$

or

$$\omega = 2p - \kappa - \frac{1}{\omega} - 2s, \quad p = ns - r.$$

II. If the modular equation belong to Class III., we obtain values for $\sqrt{k\kappa}$ or $\sqrt[4]{k\kappa}$ corresponding to

$$\omega = 4p - \kappa - \frac{1}{\omega} - 2s.$$

As before,

$$\phi^4(\Omega) = \psi^4(\omega) = \phi^4\left(-\frac{1}{\omega} - 2s\right);$$

therefore

$$\frac{\omega - 2p}{\kappa} = -\frac{1}{\omega} - 2s,$$

or

$$\omega = 4p + \sqrt{-(n - 4p^2)}, \quad p = ns - r.$$

III. If the modular equation belong to Class IV., we obtain values of $2\kappa\kappa'$ corresponding to

$$\omega = 2p + \sqrt{-(n - 4p^2)}.$$

$$\phi^4(\Omega) = \psi^4(\omega) = \phi^4\left(-\frac{1}{\omega} + 4s\right);$$

therefore

$$\omega = 2p + \sqrt{-(n - 4p^2)}, \quad p = ns - r.$$

In all these cases, when any extraneous factors appear in the result of putting $\sqrt{\lambda} = \sqrt[4]{\kappa}$, $\sqrt[4]{\lambda'} = \sqrt[4]{\kappa}$, they always appear squared. Thus, for example, in the case of $n = 17$, the equation for

$$z = \sqrt[3]{2\kappa\kappa'}$$

is

$$(z-1)(z^3-3z-1)^2(z^4-5z^3+4z^2-5z+1) = 0,$$

in which the factor $(z^3-3z-1)^2$ corresponds to

$$\omega = 2 + \sqrt{-13}.$$

By these theorems I was led to recognise that the equations for

$$\omega = \sqrt{-71} \quad \text{and} \quad \omega = \sqrt{-79}$$

(see Greenhill, *Proceedings*, Vol. xix., p. 323) ought to contain the

factors $(z+1)^2$ and $(z^2-z-1)^2$ corresponding to

$$\omega = 8+\sqrt{-7} \text{ and } \omega = 8+\sqrt{-15} \text{ respectively.}$$

The equation of the 71st order (new form) reduces therefore to

$$(z+1)^2\,(z^7+z^6-z^5-z^4-z^3+z^2+2z-1) = 0,$$

$$z = \sqrt[12]{16\kappa\kappa'};$$

and the equation of 79th order to

$$(z^2-z-1)^2\,(z^5-4z^4+4z^3-5z^2+12z-1) = 0,$$

$$z = \sqrt[4]{16\kappa\kappa'}.$$

From the quintic factor corresponding to $\omega = \sqrt{-79}$, there is no difficulty in deducing

$$\zeta^5-\zeta^4+\zeta^3-2\zeta^2+3\zeta-1 = 0,$$

$$\zeta = \sqrt[12]{16\kappa\kappa'} = z^3.$$

8. The theorems I., II., III., just proved, being of great importance in the formation of the modular equations, I thought it worth while to give separate proofs of them. They are, however, particular cases of the following :—

If in any modular equation we put

$$\sqrt[4]{\lambda} = e^{\frac{1}{4}(\alpha+\beta)}\sqrt[4]{\kappa'}, \quad \sqrt[4]{\lambda'} = \sqrt[4]{\kappa},$$

we obtain values of z corresponding to

$$\omega = ns-8r+\sqrt{-\left[n-(ns-8r)^2\right]},$$

and s and r can always be chosen so that ns−8r may be equal to any integer.

The proof is exactly the same as before:

$$\sqrt[4]{\lambda} = e^{\frac{1}{4}(\alpha\omega+\beta)}\sqrt[4]{\kappa'}$$

gives $$\phi(\Omega) = e^{\frac{1}{4}(\alpha\omega+\beta)}\phi\left(-\frac{1}{\omega}\right) = \phi\left(2s-\frac{1}{\omega}\right);$$

therefore $$\Omega = \frac{\omega+16r}{n} = 2s-\frac{1}{\omega};$$

and therefore $$\omega = ns-8r+\sqrt{-\left[n-(ns-8r)^2\right]}.$$

For example, from the modular equation of the 47th order

$$P^3-4Q-(4R)^{\frac{1}{2}}P-2(4R)^{\frac{1}{2}} = 0$$

[illegible] the values of α [illegible] corresponding to

$$\alpha = \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad , s = 1$$

$$\cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad , s = 2$$

$$\cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad \cdot \overline{\quad} \quad , s = 3 \qquad \cdot \overline{\quad} \quad , s = 4 .$$

[illegible] the [illegible] corresponding to these values of α within the same [illegible]

[illegible] the [illegible] equation belong to Class III. or Class IV., then the [illegible]

$$\cdot \quad = \quad \cdot \quad \cdot \quad \text{or} \quad = \quad \cdot$$

and [illegible] of these the values α corresponding to $s = 0, 4$: $\alpha = \cdot \quad \alpha = \cdot$ and [illegible] the latter the values corresponding to $\alpha = \cdot \quad \alpha = \cdot$ [illegible] respectively occur together.

By means of this last theorem we have a series of independent [illegible] which will [illegible] determine the coefficients [illegible] we have to do is to compare the result of elimination with the product of the factors (when [illegible] and when the modular equation can be readily [illegible] the former method is to be preferred) to obtain equations for all values of α [illegible] by the formula

$$\alpha = p - \cdot \overline{\quad} \cdot - p .$$

Modular equation for $n = 73$ gives the moduli for

$$\alpha - \sqrt{\quad} 73, \quad \sqrt{-72}, \quad \sqrt{-65}, \quad \sqrt{-60}, \quad \sqrt{-63}, \quad \sqrt{-54},$$
$$\sqrt{-43}, \quad \sqrt{-39}, \quad \sqrt{-15};$$

modular equation for $n = 71$ gives the moduli for

$$\alpha - \sqrt{\quad} 71, \quad \sqrt{-70}, \quad \sqrt{-67}, \quad \sqrt{-62}, \quad \sqrt{-55}, \quad \sqrt{-46},$$
$$\sqrt{-35}, \quad \sqrt{-22}, \quad \sqrt{-7};$$

modular equation for $n = 47$ gives the moduli for

$$\alpha - \sqrt{\quad} 47, \quad \sqrt{\quad} 46, \quad \sqrt{-43}, \quad \sqrt{-38}, \quad \sqrt{-31}, \quad \sqrt{-22}, \quad \sqrt{-11};$$

and from the modular equation of the 143rd order, which is of the form

$$a(16)^{12} + b(16)^{11} P + \ldots + (P^2 - 4Q)^{12} P = 0;$$

we can determine the moduli for

$$\omega = \sqrt{-143},\ \sqrt{-142},\ \sqrt{-139},\ \sqrt{-134},\ \sqrt{-127},\ \sqrt{-118},$$
$$\sqrt{-107},\ \sqrt{-94},\ \sqrt{-79},\ \sqrt{-62},\ \sqrt{-43},\ \sqrt{-22},$$

in which it will be noticed that the four numbers 143, 139, 127, and 107 appear, and therefore the moduli for these values of Δ which are yet unknown will at once result from the calculation of the modular equation.

9. *Modular Equation of the* 29*th order.*—The theorems we have been discussing enable us without much difficulty to determine completely the modular equation for $n = 29$. Its form is

$$AR^{\frac{1}{2}} + BR^{\frac{1}{2}}P + R(CP^2 + DQ) + R^{\frac{1}{2}}(EP^3 + FPQ)$$
$$+ R^{\frac{1}{2}}(GP^4 + HP^2Q + IQ^2) + P^5 = 0;$$

and if in this we put $\lambda = \kappa'$, $\lambda' = \kappa$, we must get an equation for $z = \sqrt[3]{2\kappa\kappa'}$ corresponding to

$$\omega = 2p + \sqrt{-(29 - 4p^2)},$$

or
$$\omega = \sqrt{-29},\ 2 + \sqrt{-25},\ 4 + \sqrt{-13};$$

but the equations in z corresponding to

$$\omega = 2 + \sqrt{-25},\quad 4 + \sqrt{-13}$$

are
$$z^3 + 7z + 1,\quad z^3 + 3z - 1;$$

and therefore the result of replacing

$$P,\ Q,\ R^{\frac{1}{2}} \text{ by } z^3 - 1,\ \frac{z^6 - 4z^3}{4},\ -\frac{z^3}{\sqrt[3]{4}}$$

is

$$(z+1)(z^3+7z+1)^2(z^3+3z-1)^2(z^6+az^5+bz^4+cz^3-bz^2+az-1) = 0,$$

where the sextic factor corresponds to $\omega = \sqrt{-29}$.

Now to find the sextic factor it will be necessary to determine the coefficients G, H, I, which may be readily done by the "q formulæ," as follows:—

As far as these coefficients are concerned, we may replace P, Q, $R^{\frac{1}{2}}$ by $\kappa' - 1$, $-\kappa'$, $-\sqrt[3]{\kappa\kappa'\lambda}$; for the terms omitted involve powers of q higher than those required for the determination of G, H, I: the first four terms may therefore be written

$$(\kappa'-1)^5 - \sqrt[3]{\kappa\kappa'\lambda}\left[G(\kappa'-1)^4 - H\kappa'(\kappa'-1)^2 + I\kappa'^2\right] \ldots\ldots = 0,$$

and, dividing by $(\kappa'-1)^4$, we have

$$\frac{1}{\sqrt[3]{\lambda}}\,\frac{1-\kappa'}{\sqrt[3]{\kappa\kappa'}}+G-H\frac{\kappa'}{(1-\kappa')^3}+I\frac{\kappa'^2}{(1-\kappa')^6}\ldots\ldots=0,$$

in which we may take $\sqrt[3]{\lambda}=\sqrt[3]{4}q^{\frac{1}{3}}$; and therefore the coefficients of $\frac{1}{q^4}$, $\frac{1}{q^3}$, and absolute term in

$$\frac{2\sqrt[3]{4}}{q^4}(1+q^3+3q^6+\ldots)^3+G-\frac{H}{4^3q^3}(1-8q^3+\ldots)+\frac{I}{4^6q^4}(1-8q^3+209^6\ldots)^3$$

must vanish.

This at once gives

$$I=-2^{13}\sqrt[3]{4},\quad H=2^8.3^3\sqrt[3]{4},\quad G=-96\sqrt[3]{4},$$

and leads immediately to the sextic factor above.

When P, Q, R^1 are replaced by

$$z^3-1,\quad \frac{z^6-4z^3}{4},\quad -\frac{z^3}{\sqrt[3]{4}},$$

the modular equation becomes

$$z^{15}+30z^{14}+\ldots \text{ unknown terms } +5z^3+94z^3-1=0,$$

which must be identical with

$$(z+1)(z^2+7z+1)^2(z^2+3z-1)^2(z^6+az^5+bz^4+cz^3-bz^2+az-1);$$

equating coefficients of z, z^2, z^3 we have immediately

$$a=9,\ b=5,\ c=-2,$$

and therefore

$$z^6+9z^5+5z^4-2z^3-5z^2+9z-1=0,\quad \omega=\sqrt{-29}.$$

The identification of the result of the above substitution in the modular equation with the product of factors enables us to write the equation as follows:—

$$-2^5.15^2(4R)^{\frac{1}{2}}+2^4.471P(4R)^{\frac{1}{2}}+2^3(4R)(2^7.15Q-745P^2)$$
$$+2^3(4R)^{\frac{1}{2}}(505P^3-2^7.13PQ)+(4R)^{\frac{1}{2}}(-94P^4+2^8.3^3P^2Q-2^{13}Q^2)+P^6=0.*$$

The sextic factor was, I believe, first discovered by Professor A. G. Greenhill, by combining the equations of the 2nd and 19th orders, and was given by him in the form

$$z^{18}+588z^{15}-979z^{12}+1960z^9+979z^6+588z^3-1=0.$$

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* See Art. 28 for simplified form of this equation.

10. *Modular Equation of the 37th order.*—This is of the 19th degree in $\kappa\lambda$, $\kappa'\lambda'$, and is not reducible to a simplified form. I have consequently made no attempt to form it. It would be much simpler to form the equation in terms of the quantities $\sqrt[4]{\lambda\kappa'}$, $\sqrt[4]{\lambda\kappa'}$, or in terms of Jacobi's u and v.

11. *The Modular Equation for* $n = 41$ can be written

$$aR^{\frac{1}{2}} + bR^2P + R^{\frac{1}{2}}(cP^2 + dQ) + R^{\frac{1}{2}}(eP^3 + fPQ) + R(gP^4 + hP^2Q + iQ^2)$$
$$+ R^{\frac{1}{2}}(jP^5 + lP^3Q + mPQ^2) + R^{\frac{1}{2}}(nP^6 + pP^4Q + rP^2Q^2 + sQ^3) + P^7 = 0;$$

and when we put $\lambda = \kappa'$, $\lambda' = \kappa$, and $z = \sqrt[3]{2\kappa\kappa'}$, we get for z the equation

$$(z-1)(z^2-z-1)^2(z^2-7z+1)^2(z^2-12z-1)^2$$
$$\left[(z^4+7z^3+14z^2+7z+1)^2 - z(5z^3+12z^2+12z+5)^2\right] = 0,$$

in which the factors which appear correspond to $\omega = 6+\sqrt{-5}$, $4+\sqrt{-25}$, $2+\sqrt{-37}$, and $\sqrt{-41}$, respectively. The last factor is due to Weber, and is given by Greenhill in the article already referred to (p. 363). I have not calculated the coefficients of this equation, but if required at any future time they may be obtained by identifying the result of replacing P, Q, $\sqrt[3]{4R}$ by z^3, $\frac{z^6-4z^3}{4}$, $-z^2$. The seven coefficients j, l, m, n, p, r, s are very readily indeed obtained by the method adopted in the case of $n = 29$.

12. The only other equation of this class which I shall attempt to form is that for $n = 53$, and my object in doing so is to obtain the sextic equation for $\sqrt[3]{2\kappa\kappa'}$ corresponding to $\omega = \sqrt{-53}$. It is comparatively easy, as I shall afterwards explain, to obtain the equation for $z^3 = 2\kappa\kappa'$; but considerable difficulty is experienced in throwing that result into the form $U^3 + V^3 + W^3 - 3UVW = 0$, and thus obtaining an equation for $\sqrt[3]{2\kappa\kappa'}$.

The modular equation in this case may be written

$$P^9 + (4R)^{\frac{1}{2}}(AQ^4 + BQ^3P^2 + CQ^2P^4 + DQP^6 + EP^8) + \ldots = 0;$$

and, as in the case of $n = 29$, we may for the determination of A, B, C, D, E replace P, Q, $\sqrt[3]{4R}$ by $\kappa'-1$, $-\kappa'$, $-\sqrt[3]{4\kappa\kappa'\lambda}$; and therefore, on dividing by $(\kappa'-1)^8$, we obtain

$$\frac{1}{\sqrt[3]{\lambda}}\,\frac{1-\kappa'}{\sqrt[3]{4\kappa\kappa'}} + \left[\frac{A}{4^4}\,\frac{1}{\phi^{32}(2\omega+1)} + \frac{B}{4^3}\,\frac{1}{\phi^{24}(2\omega+1)} + \frac{C}{4^2}\,\frac{1}{\phi^{16}(2\omega+1)}\right.$$
$$\left. + \frac{D}{4}\,\frac{1}{\phi^{8}(2\omega+1)} + E\right] + \ldots = 0.$$

But

$$\frac{1-\kappa'}{\sqrt[3]{4\kappa\kappa'}} = 2\sqrt[3]{4}q^{\frac{1}{3}}(1+q^2+3q^4+4q^6+7q^8+10q^{10}+\ldots)$$

$$-\frac{4\kappa'}{(1-\kappa')^2} = \frac{1}{\phi^8(2\omega+1)} = -\frac{1}{16q^2}(1-8q^2+20q^4\ldots-62q^8\ \ldots),$$

q^6 and q^{10} being absent,

$$\sqrt[3]{\lambda} = \sqrt[3]{4}\,q^{\frac{1}{3}};$$

hence we have, in

$$\frac{2\sqrt[3]{4}}{q^{\frac{8}{3}}}(1+q^2+3q^4+4q^6+7q^8+10q^{10})^2 + \frac{A}{4^{12}q^8}\left[1-8q^2+20q^4\ldots-62q^8\ldots\right]^4$$

$$-\frac{B}{4^9q^6}\left[1-8q^2+20q^4\ldots\right]^3$$

$$\frac{C}{4^6q^4}\left[1-8q^2+20q^4\ldots\right]^2$$

$$-\frac{D}{4^3q^2}\left[1-8q^2\right]$$

$$+E,$$

to find the coefficients so that all the terms from $\frac{1}{q^{\frac{8}{3}}}$ to the absolute term may vanish, and it will be easily seen that the coefficients of A, B, C, D have only to be expanded as far as q^8, q^6, q^4, q^2, respectively.

It seems unnecessary to actually write down the numerical work, as it is neither difficult nor laborious. I shall content myself with giving the values of A, B, C, D, E. They are:—

$$A = -2^{25}\cdot\sqrt[3]{4},\quad B = 2^{20}\cdot 17\sqrt[3]{4},\quad C = -2^{13}\cdot 359\sqrt[3]{4},$$

$$D = 2^8\cdot 579\sqrt[3]{4},\quad E = -798\sqrt[3]{4};$$

and the modular equation for $n = 53$ becomes

$$P^{54}+2\sqrt[3]{4R}\,(-2^{24}Q^4+2^{19}\cdot 17Q^3P^2-2^{12}\cdot 359Q^2P^4+2^7\cdot 579QP^6-399P^8)^3$$

$$+\ldots\ = 0.$$

On making the usual substitutions for determining $\sqrt[3]{2\kappa\kappa'}$, we obtain

$$z^{27}+\text{ unknown terms }+\ldots+798z^3-1 = 0,$$

where it is important to notice that z^3 is absent.

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* See Art. 29 for the complete equation.

Identifying the coefficients of z, z^2, z^3 in this with

$$(z+1)(z^4+18z^3-29z^2+18z+1)^2\,(z^2+12z-1)^2\,(z^4+5z^3+4z^2+5z+1)^2$$
$$(z^6+az^5+bz^4+cz^3-bz^2+az-1),$$

where the several factors correspond to

$$\omega = 2+\sqrt{-49},\quad 4+\sqrt{-37},\quad 6+\sqrt{-17},\quad \sqrt{-53},$$

we easily find $a = 23$, $b = 12$. $c = 75$; and the equation for $\sqrt[3]{2\kappa\kappa'} = z$, corresponding to $\omega = \sqrt{-53}$, is

$$z^6+23z^5+12z^4+75z^3-12z^2+23z-1 = 0.$$

As in the case of $n = 29$, Prof. Greenhill had communicated to me the equation for z^3, but the coefficients were so large that I chose to obtain the equation for $\sqrt[3]{2\kappa\kappa'}$ directly from the modular equation rather than endeavour to reduce his result to the form

$$U^3+V^3+W^3-3UVW = 0.$$

Prof. Greenhill has also sent me the equation for $z^3 = 2\kappa\kappa'$ where $\omega = \sqrt{-61}$, viz.,

$$z^{18}+26604z^{15}-37587z^{12}-6552z^9+37587z^6+26604z-1 = 0,$$

obtained by combining the equations of the 2nd and 31st order; and has succeeded in obtaining from it the equation

$$z^6+30z^5+6z^4+48z^3-6z^2+30z-1 = 0.$$

The direct determination of the equation for z without having to pass from the equation for z^3 forms a very beautiful example in the theory of the Complex Multiplication Moduli, which we shall afterwards solve by means of a new form for the equation of the 7th order.

13. The next equation I shall endeavour to obtain is that for $n = 43$. It belongs to class III., and is of the 11th degree in $\sqrt{\kappa\lambda}$, $\sqrt{\kappa'\lambda'}$, -1. It may therefore be written

$$P^{11}+R\,(AQ^4+BQ^3P^3+CQ^2P^4+DQP^6+EP^8)$$
$$+R^2\,(FQ^2P+GQP^3+HP^5)+R^3\,(IQ+JP^2) = 0,$$

and we shall obtain the coefficients A, B, C, D, E by a process similar to that in $n = 29$. For the determination of these terms, P, Q, R may be replaced by $\sqrt{\kappa'}-1$, $-\sqrt{\kappa'}$, $-\sqrt{\kappa\kappa'\lambda}$, and the modular

equation may be written

$$\frac{1}{\sqrt{\lambda}}\frac{1-\sqrt{\kappa'}}{\sqrt{\kappa}}+A\left[\frac{\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2}\right]^5-B\left[\frac{\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2}\right]^4$$

$$+C\left[\frac{\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2}\right]^3-D\left[\frac{\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2}\right]^2+E\frac{\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2}, \text{ \&c...} = 0.$$

But $\dfrac{1-\sqrt{\kappa'}}{\sqrt{\kappa}} = \phi(2\omega)\,\varphi(4\omega)$ and $\dfrac{4\sqrt{\kappa'}}{(1-\sqrt{\kappa'})^2} = \dfrac{1}{\phi^4(4\omega)} - 1$

$$= 2q^{\frac{1}{2}}(1-q^2+q^4-2q^6+4q^8\ldots) \qquad = \frac{1-4q^2+4q^4+2q^8}{4q^2}.$$

Hence we easily find, on the substitution of these expressions, that

$$A = -4^{10}, \quad B = 4^8 . 19, \quad C = -4^6 . 125, \quad D = 4^4 . 330,$$
$$E = -4^2 . 286,$$

and the modular equation becomes

$$P^{11}+R\,(-4^{10}Q^4+4^8 . 19\,Q^3P^2-4^6 . 125\,Q^2P^4+4^4 . 330\,QP^6-4^2 . 286\,P^8)$$
$$+E^2\,(FQ^3P+GQP^3+HP^5)+E^3\,(IQ+JP^3) = 0.$$

The result of putting

$$\sqrt{\lambda} = \sqrt{\kappa'}, \quad \sqrt{\lambda'} = \sqrt{\kappa}, \quad z = 2\sqrt{\kappa\kappa'},$$

or, in other words,

$$P = z-1, \quad Q = \frac{z^2-4z}{4}, \quad R = -\frac{z^3}{4},$$

is $\left[(z^3+3z^2+15z+1)\,(z+1)\right]^2$ (cubic in z),

in which the factors correspond to

$$\omega = 4+\sqrt{-27} \text{ and } \sqrt{-43},$$

respectively; and, as in class IV., the irrelevant factors always appear squared. On comparing this equation with the modular equation, the cubic in z is at once seen to be

$$z^3+5z^2+43z-1 = 0,^*$$

which may be written

$$z+2\sqrt[3]{4}\,z^{\frac{1}{3}}+\sqrt[3]{16}\,z^{\frac{1}{3}}-1 = 0,$$

or

$$z^{\frac{1}{3}}+\sqrt[3]{4}\,z^{\frac{1}{3}}-1 = 0.$$

* It has been already mentioned that this factor can be immediately determined from $n = 47$.

The remaining coefficients can at once be found by equating the coefficients of powers of z in the modular equation, and

$$(z+1)^2\,(z^3+3z^2+15z+1)^2\,(z^3+5z^2+43z-1).$$

14. *Modular Equation of the 59th order.*—This is the last modular equation which I shall attempt to calculate. Here the equation takes a form similar to that of $n = 29$, and may therefore be written

$$AR^5+BR^4P+R\,(CP^3+DQ)+R^2\,(EP^3+FPQ)$$
$$+R^3\,(GP^4+HP^2Q+IQ^2)+P^5=0,$$

where

$$P=\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1,\quad Q=\sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'},\quad R=-\sqrt{\kappa\lambda\kappa'\lambda'}.$$

As in previous cases, the values of G, H, I, E, F are very readily determined by the "q formulæ." Such a process, however, is not necessary; for, as I shall afterwards prove, the equation for $\zeta=\sqrt[12]{4\kappa\kappa'}$, when $\omega=\sqrt{-59}$, is

$$\zeta^{12}-2\sqrt[3]{2}\,\zeta^8+\sqrt[3]{4}\,\zeta^7-\zeta^6+2\sqrt[3]{2}\,\zeta^5-\sqrt[3]{4}\,\zeta^4+\zeta^3-2\sqrt[3]{2}\,\zeta^2+2\sqrt[3]{4}\,\zeta-1=0;$$

and, therefore, the equation for $z=\sqrt[6]{4\kappa\kappa'}$ is

$$z^{12}-2\sqrt[3]{4}\,z^8+2\sqrt[3]{2}\,z^7+z^6+2\sqrt[3]{2}\,z^4+7z^3-2\sqrt[3]{4}\,z^2+4\sqrt[3]{2}\,z-1=0.$$

Now, when P, Q, R are replaced by z^3-1, $\frac{z^6-4z^3}{4}$, $-\frac{z^3}{\sqrt[3]{4}}$, the factors correspond to $\omega=\sqrt{-59}$, $4+\sqrt{-43}$, and the result, therefore, is $(z^3-2\sqrt[3]{4}\,z^2+2\sqrt[3]{2}\,z+1)^2$ (factor in z just obtained). If we equate coefficients, we without much difficulty determine A, B, C, ... I. The result is

$$-8\,(16R)^{\frac{5}{2}}-4P\,(16R)^2-16R\,(42P^2+96Q)+(16R)^{\frac{3}{2}}P\,(19P^2-16Q)$$
$$-(16R)^1\,(14P^4-144P^2Q+256Q^2)+P^5=0.$$

15. *Relation between Jacobian and $\kappa\lambda$, $\kappa'\lambda'$ Modular Equations.*—It may not be without interest to show the connexion between the equations in $\kappa\lambda$, $\kappa'\lambda'$ and the Jacobian Modular Equations.

It is easily proved that

$$\frac{R}{P^3}\equiv-\frac{\kappa\lambda\kappa'\lambda'}{(\kappa\lambda+\kappa'\lambda'-1)^3}=\frac{\phi^3(2\omega)\,\phi^3(2n\omega)\,\psi^3(2\omega)\,\psi^3(2n\omega)}{2\left[\phi^3(2\omega)-\phi^3(2n\omega\right]^6},$$

$$\frac{Q}{P^2}\equiv\frac{\kappa\lambda\kappa'\lambda'-\kappa\lambda-\kappa'\lambda'}{(\kappa\lambda+\kappa'\lambda'-1)^3}$$
$$=-\frac{8\phi^3(2\omega)\,\phi^3(2n\omega)\left[\phi^4(2\omega)+\phi^4(2n\omega)\right]+\psi^3(2\omega)\,\psi^3(2n\omega)}{4\left[\phi^3(2\omega)-\phi^3(2n\omega)\right]^4};$$

hence, if we denote $\phi(2\omega)$, $\phi(2n\omega)$ by u, v, we have

$$\rho P = (u^2-v^2)^2; \quad -4\rho^2 Q = 8u^2v^2(u^4+v^4)+(u'v')^8;$$

$$\rho(2R)^{\frac{1}{2}} = (uvu'v')^{\frac{1}{2}}(u'v')^2;$$

and, if $F(u, v)$ be the Jacobian modular equation, the result of the above substitution is, therefore, the product

$$F(u, v) \cdot F(u, -v) = 0.$$

In such cases as $n = 37, 61$, it is therefore easier to find the Jacobian equation than that of the class principally dealt with in this paper.

Complex Multiplication Moduli.

16. I shall now explain some methods for obtaining the values of the moduli for which $\omega = \sqrt{-n}$, where n is any integer. The function of the moduli which we determine is

$\sqrt[4]{16\kappa\kappa'}$, when n belongs to class I. or II.;

$\sqrt{4\kappa\kappa'}$, when n belongs to class III.;

$2\kappa\kappa'$, when n belongs to class IV.;

and we have seen that, when $n+1$ is divisible by 3, the equation reduces to one for determining $\sqrt[12]{16\kappa\kappa'}$, $\sqrt[6]{4\kappa\kappa'}$, $\sqrt[3]{2\kappa\kappa'}$, according to the class to which n belongs. It can be readily seen, however, that whatever be the value of n, unless it be divisible by 3, the equation for determining the moduli can always be reduced to one for

$\sqrt[12]{16\kappa\kappa'}$, in classes I. and II.;

$\sqrt[12]{4\kappa\kappa'}$, in class III.;

$\sqrt[3]{2\kappa\kappa'}$, in class IV.;

and, in the last case, the equation is always reciprocal.

The reason of this is very simple. We saw that from any modular equation of order n can be determined the moduli corresponding to

$$\omega = \sqrt{-(n-4p^2)};$$

hence, if n belongs to class I., II., or III., we find some value of p for which $n+4p^2$ belongs to class I. or II., is prime, and such that $n+4p^2+1$ is divisible by 3; and therefore the modular equation of order $n+4p^2$ reducible to form

$$U^3+V^3+W^3-3UVW = 0.$$

This being possible, we can always obtain equations to determine $\sqrt[12]{16\kappa\kappa'}$ in the first three classes.

If n belongs to class IV., so also does $n+4p^2$, and in this case we can therefore reduce our equations so as to determine $\sqrt[3]{2\kappa\kappa'}$.

A few examples will make the theory more clear. The results for

$$\omega = \sqrt{-43},\quad \sqrt{-59},\quad \sqrt{-67},\quad \sqrt{-79},\quad \sqrt{-95}$$

can be deduced from the modular equation for orders 47, 383, 71, 143, 239, respectively, each of which belongs either to class I. or II., and $n+1$ is divisible by 3. Again, the results for

$$\omega = \sqrt{-13},\quad \sqrt{-25},\quad \sqrt{-37},\quad \sqrt{-61}$$

can be obtained from the modular equations of the 17th, 29th, 41st and 257th orders, respectively, and therefore in each of these cases we may expect equations for $\sqrt[3]{2\kappa\kappa'}$.

17. Having explained the nature of the simplest function of $\kappa\kappa'$ in terms of which the complex multiplication equations can be expressed, I proceed at once to show how in practice they are to be determined.

If the modular equations of the m^{th} and n^{th} orders be written

$$F_m(\Omega_1, \omega) = 0,\quad F_n(\Omega, \Omega_1) = 0,$$

then
$$\Omega_1 = \frac{\omega + 16r}{m},\quad \Omega = \frac{\Omega_1 + 16s}{n}.$$

And if we assume $\Omega = -\frac{1}{\omega} + 2t$, then we have

$$\omega = mnt - 8r - 8ms + \sqrt{-\left[mn - (mnt - 8r - 8ms)^2\right]};$$

and therefore, from the modular equations of the m^{th} and n^{th} orders, it is possible to deduce the values of the moduli for all those values of ω given by the equation

$$\omega = p + \sqrt{-(mn - p^2)}.$$

A very beautiful set of examples is the deduction of the equations for $\sqrt[3]{2\kappa\kappa'}$ or $\sqrt[6]{2\kappa\kappa'}$, where $\omega = \sqrt{-77},\ \sqrt{-73},\ \sqrt{-61},\ \sqrt{-41}$, and $\sqrt{-13}$, from the modular equations of the 11th and 7th orders.

Modular Equation of 7th Order.

18. The usual form in which this equation is written is due to Gutzlaff, and is
$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} - 1 = 0.$$

I recently found that it could be written in the form

$$\kappa\lambda'+\kappa'\lambda = \sqrt{2}\sqrt[12]{4\kappa\lambda\kappa'\lambda'}\,(\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'}),$$

which can be at once verified by means of the "q formulæ."

It is more convenient, however, to deduce it directly from

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = 1.$$

Squaring twice, we get

$$\kappa\lambda+\kappa'\lambda' = 1-2\sqrt{2}t^3+t^6 \text{ if } t = \sqrt[12]{4\kappa\lambda\kappa'\lambda'},$$

and

$$(\kappa\lambda'+\kappa'\lambda)^2 = 4\sqrt{2}t^3-10t^6+4\sqrt{2}t^9;$$

therefore

$$(2\kappa\kappa'+2\lambda\lambda')^2 = t^8\,(2\sqrt{2}-5t^3+2\sqrt{2}t^6)(2\sqrt{2}-8t^3+2\sqrt{2}t^6)^2.$$

Putting

$$(2\kappa\kappa')^{\frac{1}{4}}+(2\lambda\lambda')^{\frac{1}{4}} = \Theta,$$

and

$$\sqrt{2\sqrt{2}-5t^3+2\sqrt{2}t^6} = \Phi,$$

we have

$$\Theta^8-3t^4\Theta = t^{\frac{1}{2}}\Phi\,(\Phi^2-3t^3) = t^{\frac{1}{2}}\Phi^3-3t^4\,.\,t^{\frac{1}{2}}\Phi,$$

and therefore

$$\Theta = t^{\frac{1}{2}}\Phi,$$

or

$$\sqrt{2}\,t\,(\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'}) = \kappa\lambda'+\kappa'\lambda.$$

19. *Moduli corresponding to*

$$\omega = \sqrt{-73}, \quad \sqrt{-41}.$$

If we take $m = 11$, $n = 7$ in the general theory, then, in order that $mnt-8r-8ms = 2$, we must have $t = 2$, and with $t = 2$ we have

$$mnt-8r-8ms = 2 \text{ or } -6,$$

and therefore

$$\omega = 2+\sqrt{-73}, \quad -6+\sqrt{-41}.$$

Since $t = 2$, the modular equations must be written

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{4\kappa\lambda\kappa'\lambda'} = 1 \quad \ldots\ldots(1),$$

$$e^{\frac{1}{6}\pi i}\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'} = 1 \quad \ldots\ldots(2),$$

and, if we put $\sqrt[12]{4\kappa\lambda\kappa'\lambda'} = re^{\frac{1}{6}\pi i}$, we have from the former

$$\kappa\lambda+\kappa'\lambda' = 1+4r^2+4r^4+r^6$$

and

$$\kappa\lambda'+\kappa'\lambda = \sqrt{-2}\,(2r+3r^3+2r^5) \quad \ldots\ldots(3),$$

and from the latter

$$\kappa'\lambda+\kappa\lambda' = 1+2\sqrt{2}r^3+r^6$$

and

$$\kappa\lambda+\kappa'\lambda' = \sqrt{-1}\,\sqrt{4\sqrt{2}r^3+10r^6+4\sqrt{2}r^9} \quad \ldots\ldots(4).$$

Proceeding as in the previous paragraph, we deduce

$$\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'}=-\sqrt{-1}\,r^4\sqrt{2\sqrt{2}+5r^2+2\sqrt{2}\,r^6}$$

$$=-\frac{1}{r\sqrt{2}}(\kappa\lambda+\kappa'\lambda')$$

$$=-\frac{1}{r\sqrt{2}}(1+4r^2+4r^4+r^6) \quad \ldots\ldots\ldots\ldots\ldots \quad (5).$$

Now, from (3) and (4) we have for r the equation

$$1+2\sqrt{2}r^3+r^6-\sqrt{-2}\,(2r+3r^3+2r^5)=0,$$

which, on putting $r+\frac{1}{r}=\frac{\rho}{\sqrt{2}}$, becomes

$$\left[\rho-(1+3\sqrt{-1})\right]\left[\rho^2+(1-\sqrt{-1})\,\rho-2(1-\sqrt{-1})\right]=0,$$

the factors of which correspond respectively to

$$\omega=2+\sqrt{-73}, \quad -6+\sqrt{-41}.$$

Taking $\rho=1+3\sqrt{-1}$, we have

$$\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'}=-\frac{1}{r\sqrt{2}}(1+4r^2+4r^4+r^6),$$

and, since

$$r^4=\sqrt[3]{4\kappa\kappa'\lambda\lambda'},$$

$$\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\frac{1}{\sqrt[3]{2\lambda\lambda'}}=-\frac{1}{r^5\sqrt{2}}(1+4r^2+4r^4+r^6);$$

therefore

$$\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'}+\frac{1}{\sqrt[3]{2\lambda\lambda'}}+\sqrt[3]{2\lambda\lambda'}$$

$$=-\frac{1}{\sqrt{2}}\left[r^5+\frac{1}{r^5}+4\left(r^3+\frac{1}{r^3}\right)+5\left(r+\frac{1}{r}\right)\right]$$

$$=-45,$$

and

$$\left(\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'}\right)\left(\frac{1}{\sqrt[3]{2\lambda\lambda'}}+\sqrt[3]{2\lambda\lambda'}\right)=50.$$

Therefore $\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'}$ is a root of the quadratic

$$z^2+45z+50=0.$$

In this result we must recollect that

$$\sqrt[24]{\kappa}=\phi(2+\sqrt{-73})=e^{\frac{\pi}{24}\sqrt{-73}}\,\phi(\sqrt{-73});$$

hence, if $$x = \sqrt[6]{2\phi^4(\sqrt{-73})\,\psi^4(\sqrt{-73})},$$

then $$z = -\left(x^3 + \frac{1}{x^3}\right),$$

and therefore we have, for $x + \frac{1}{x} = \zeta$, the equation

$$(\zeta^3-2)^2 - 45(\zeta^2-2) + 50 = 0,$$

or $$\zeta^6 - 49\zeta^2 + 144 = 0,$$

which breaks up into the factors

$$(\zeta^3-5\zeta-12)(\zeta^3+5\zeta-12) = 0,$$

giving $$\sqrt[6]{2\kappa\kappa'} + \frac{1}{\sqrt[6]{2\kappa\kappa'}} = \frac{5+\sqrt{73}}{2},$$*

where $$\sqrt[6]{\kappa} = \phi\sqrt{-73}.$$

(See Greenhill, *Proceedings*, Vol. XIX., p. 363.)

The factor

$$\rho^3 + \rho(1-\sqrt{-1}) - 2(1-\sqrt{-1}) = 0,$$

corresponding to

$$\omega = \sqrt{-41},$$

may be similarly dealt with. We have

$$\sqrt[3]{2\kappa\kappa'} + \sqrt[3]{2\lambda\lambda'} = -\frac{1}{\tau\sqrt{2}}(1+4\tau^2+4\tau^4+\tau^6);$$

---

* This example might have been more easily solved by taking into account that

$$\sqrt[3]{2\kappa\kappa'} + \sqrt[3]{2\lambda\lambda'} = -\sqrt{-1}\,\tau^2\sqrt{2\sqrt{2}\left(\tau^3 + \frac{1}{\tau^3}\right) + 5},$$

where the part inside the radical is $(3-6\sqrt{-1})^2$. We could thus get directly the equation for $\sqrt[6]{2\kappa\kappa'} + \frac{1}{\sqrt[6]{2\kappa\kappa'}}$. I may also notice that

$$\rho = 1 + 3\sqrt{-1}$$

gives for τ the quartic $\tau^4 - \sqrt{2}\,\tau^3 + 7\tau^2 - \sqrt{2}\,\tau + 1 = 0$;

and, if we find the equation whose roots are $\tau_2\tau_3 + \tau_1\tau_4$, $\tau_3\tau_1 + \tau_2\tau_4$, $\tau_1\tau_2 + \tau_3\tau_4$, we obtain $$(\zeta-2)(\zeta^2-5\zeta-12) = 0,$$

the quadratic factor giving the values of

$$\frac{1}{\sqrt[6]{2\kappa\kappa'}} + \sqrt[6]{2\kappa\kappa'} \text{ for } \omega = \sqrt{-73}.$$

therefore

$$\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'}+\frac{1}{\sqrt[3]{2\lambda\lambda'}}+\sqrt[3]{2\lambda\lambda'}$$

$$=-\frac{1}{\sqrt{2}}\left[r^5+\frac{1}{r^5}+4\left(r^3+\frac{1}{r^3}\right)+5\left(r+\frac{1}{r}\right)\right],$$

$$\left(\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'}\right)\left(\frac{1}{\sqrt[3]{2\lambda\lambda'}}+\sqrt[3]{2\lambda\lambda'}\right)$$

$$=r^4+\frac{1}{r^4}-2+\left(\sqrt[6]{\frac{2\kappa\kappa'}{2\lambda\lambda'}}+\sqrt[6]{\frac{2\lambda\lambda'}{2\kappa\kappa'}}\right)^2$$

$$=r^4+\frac{1}{r^4}-2-\left[2\sqrt{2}\left(r^2+\frac{1}{r^2}\right)+5\right].$$

Expressing

$$r^5+\frac{1}{r^5}+4\left(r^3+\frac{1}{r^3}\right)+5\left(r+\frac{1}{r}\right)$$

and

$$r^4+\frac{1}{r^4}-2-\left[2\sqrt{2}\left(r^2+\frac{1}{r^2}\right)+5\right]$$

in terms of ρ, and reducing by means of the equation

$$\rho^3+\rho(1-\sqrt{-1})-2(1-\sqrt{-1})=0,$$

we easily obtain

$$\frac{1}{\sqrt[3]{2\kappa\kappa'}}+\sqrt[3]{2\kappa\kappa'} \text{ and } \frac{1}{\sqrt[3]{2\lambda\lambda'}}+\sqrt[3]{2\lambda\lambda'}$$

as roots of

$$\zeta^2-\zeta\left[\frac{\rho}{2}(7+3\sqrt{-1})-3-\sqrt{-1}\right]$$

$$+\frac{\rho}{2}(13+9\sqrt{-1})-10-8\sqrt{-1}=0.$$

Solving for ρ in terms of ζ, and substituting in

$$\rho^3+(1-\sqrt{-1})(\rho-2)=0,$$

we obtain

$$2\left[\zeta^2+(3+\sqrt{-1})\zeta-10-3\sqrt{-1}\right]^2$$

$$+\left[\zeta^2-(4+2\sqrt{-1})\zeta+(3+6\sqrt{-1})\right]$$

$$\times\left[(7+3\sqrt{-1})\zeta-13-9\sqrt{-1}\right](1-\sqrt{-1})=0.$$

It is remarkable that the imaginary quantities disappear from this

and leave for $\sqrt[3]{2\kappa\kappa'}+\frac{1}{\sqrt[3]{2\kappa\kappa'}}$ the equation

$$\zeta^4+11\zeta^3-47\zeta^2+21\zeta+46=0,$$

where $$\sqrt[4]{\kappa}=\phi(-6+\sqrt{-41}).$$

If, however,

$$\sqrt[4]{\kappa}=\phi\sqrt{-41},$$

then for $$z=\sqrt[3]{2\kappa\kappa'}+\frac{1}{\sqrt[3]{2\kappa\kappa'}},$$

we have the equation

$$z^6-11z^5-47z^3-21z+46=0,$$

or $$(2z^3-11z+15)^2-41(3z-1)^2=0;$$

hence, if $$x=\sqrt[6]{2\kappa\kappa'}+\frac{1}{\sqrt[6]{2\kappa\kappa'}},$$

we have for x the equation

$$2x^4-(19+3\sqrt{41})x^2+45+7\sqrt{41}=0,$$

or $$(2x^2+7+\sqrt{41})^2-(66+10\sqrt{41})x^2,$$

or $$2x^2-(5+\sqrt{41})x+7+\sqrt{41}=0.*$$

20. *Moduli corresponding to*

$$\omega=\sqrt{-61}.$$

This is the most beautiful example of all, and can be solved very shortly as follows:—

Since in this case

$$mnt-8r-8ms=4,$$

we must take $$t=4,$$

and therefore $$\Omega=8-\frac{1}{\omega}.$$

We have, therefore, to solve between

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{4\kappa\lambda\kappa'\lambda'}=1,$$

$$-\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'}=1.$$

* This result is due to Weber. (See Greenhill, *Proceedings*, Vol. XIX., p. 363.)

Putting $\sqrt[12]{4\kappa\lambda\kappa'\lambda'} = r$, we have, by squaring,

$$\kappa\lambda + \kappa'\lambda' = 1 - 4r^2 + 4r^4 - r^6,$$

$$\kappa\lambda' + \kappa'\lambda = \pm\sqrt{2}\,(2r - 3r^3 + 2r^5),*$$

$$\kappa\lambda' + \kappa'\lambda = 1 + 2\sqrt{2}r^3 + r^6;$$

hence $$r^6 - 2\sqrt{2}r^5 + 5\sqrt{2}r^3 - 2\sqrt{2}r + 1 = 0;$$

or, denoting $r + \frac{1}{r}$ by θ,

we have $$\theta^3 - 2\sqrt{2}\theta^2 - 3\theta + 9\sqrt{2} = 0.$$

In this case, the modular equation of the 7th order may be written

$$\sqrt[3]{2\kappa\kappa'} + \sqrt[3]{2\lambda\lambda'} = -\frac{\kappa\lambda + \kappa'\lambda'}{\sqrt{2}\sqrt[12]{4\kappa\lambda\kappa'\lambda'}} = -\frac{1 - 4r^2 + 4r^4 - r^6}{r\sqrt{2}};$$

therefore

$$\frac{1}{\sqrt{2\kappa\kappa'}} + \frac{1}{\sqrt{2\lambda\lambda'}} = -\frac{1 - 4r^2 + 4r^4 - r^6}{r^5\sqrt{2}},$$

and therefore

$$\frac{1}{\sqrt[3]{2\kappa\kappa'}} - \sqrt[3]{2\kappa\kappa'} + \frac{1}{\sqrt[3]{2\lambda\lambda'}} - \sqrt[3]{2\lambda\lambda'}$$

$$= \frac{1}{\sqrt{2}}\left[r - 3\left(r + \frac{1}{r}\right) + 4\left(r^3 + \frac{1}{r^3}\right) - \left(r^5 + \frac{1}{r^5}\right)\right]$$

If we denote the three values of $\frac{1}{\sqrt[3]{2\kappa\kappa'}} - \sqrt[3]{2\kappa\kappa'}$ by z_1, z_2, z_3, and the three corresponding values of θ by θ_1, θ_2, θ_3, this last relation becomes

$$z_2 + z_3 = \frac{1}{\sqrt{2}}(-20\theta_1 + 9\theta_1^3 - \theta_1^5),$$

with similar expressions for $z_3 + z_1$ and $z_1 + z_2$; from which we easily obtain $$z_1 + z_2 + z_3 = 30,$$

and therefore

$$z_1 = 30 + \frac{1}{\sqrt{2}}(20\theta_1 - 9\theta_1^3 + \theta_1^5).$$

---

* The upper sign corresponds to $\omega = \sqrt{-61}$. The lower sign being taken, the sextic for r breaks into factors

$$\left(r + \frac{1}{r} - \sqrt{2}\right)\left(\overline{r + \frac{1}{r}}^2 - \sqrt{2}\overline{r + \frac{1}{r}} - 5\right) = 0,$$

which correspond, respectively, to $\omega = \sqrt{-13}$ and $\sqrt{-77}$. The quadratic factor leads at once to the equation

$$z^2 - z(23 + 8\sqrt{11}) + 4 = 0, \text{ where } z \equiv \frac{1}{\sqrt[3]{2\kappa\kappa'}} - \sqrt[3]{2\kappa\kappa'}.$$

But, inasmuch as the most general rational function of one root of a cubic can be reduced to the form $\frac{l\alpha+m}{l'\alpha+m'}$, we easily find

$$\theta = -3\sqrt{2}\,\frac{z-9}{z+15}.$$

Substituting this value for θ in $\theta^3-2\sqrt{2}\,\theta^2-3\theta+9\sqrt{2}=0$, we have, for $z = \frac{1}{\sqrt[4]{2\kappa\kappa'}} - \sqrt[3]{2\kappa\kappa'}$,

the equation $z^3-30z^2+9^2-18=0$,

—a result already obtained by Professor Greenhill from Hermite's Method, and communicated to me during the month of April.

21. *Moduli corresponding to*

$\omega = \sqrt{-161},\ \sqrt{-157},\ \sqrt{-145},\ \sqrt{-125},\ \sqrt{-97},\ \sqrt{-61},\ \sqrt{-17}.$

If we deal with the modular equations of the 7th and 23rd orders in a manner similar to that in which we treated the 7th and 11th, we shall derive results corresponding to

$\omega = \sqrt{-161},\ \sqrt{-157},\ \sqrt{-145},\ \sqrt{-125},\ \sqrt{-97},\ \sqrt{-61},\ \sqrt{-17}.$

Reasoning in the same manner as before, we see that when

$t=0,\ \omega = \sqrt{-161},\ 8+\sqrt{-97}\,;$

$t=2,\ \omega = 2+\sqrt{-157},\ 6+\sqrt{-125},\ 10+\sqrt{-61}\,;$

$t=4,\ \omega = 4+\sqrt{-145},\quad 12+\sqrt{-17}.$

With $t=0$, the modular equations become

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt{2}\,\sqrt[12]{4\kappa\lambda\kappa'\lambda'}=1,$$

$$\sqrt[4]{\kappa\lambda'}+\sqrt[4]{\kappa'\lambda}=1.$$

Denoting $\sqrt[12]{4\kappa\lambda\kappa'\lambda'}$ by r, Professor Greenhill has determined (*Proceedings*, Vol. XIX., p. 349) the equation for

$$r+\frac{1}{r}=\sqrt{2}\,y,$$

viz., $4y^6-32y^5+100y^4-160y^3+113y^2-24y+9=0,$

which he informs me breaks up into factors

$$(2y^2-6y+9)(2y^4-10y^3+11y^2-2y+1)=0,$$

corresponding to $\omega = 8+\sqrt{-97},\ \sqrt{-161}$, respectively.

The first factor reduces to

$$r^4-3\sqrt{2}r^3+11r^2-3\sqrt{2}r+1=0,$$

and, on finding (as in the case of $\omega=\sqrt{-73}$) the equation whose roots are $r_1r_3+r_3r_4$, $r_2r_3+r_1r_4$, $r_3r_1+r_2r_4$, we obtain

$$(\xi-2)(\xi^3-9\xi-4)=0,$$

where $$\xi=\frac{1}{\sqrt[6]{2\kappa\kappa'}}+\sqrt[6]{2\kappa\kappa'};$$

and therefore $$\frac{1}{\sqrt[6]{2\kappa\kappa'}}+\sqrt[6]{2\kappa\kappa'}=\frac{9+\sqrt{97}}{2}.$$

The second factor is treated by the same method as that adopted in the case $\omega=\sqrt{-41}$.

22. *Moduli corresponding to*

$$\omega=2+\sqrt{-157},\ -6+\sqrt{-125},\ 10+\sqrt{-61}.$$

In this case $t=2$, and therefore the modular equations are to be written

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt{2}\sqrt[12]{4\kappa\lambda\kappa'\lambda'}=1,$$

$$e^{\frac{1}{4}\pi i}\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'}=1;$$

and denoting, as before, $\sqrt[12]{4\kappa\lambda\kappa'\lambda'}$ by r, we have, for r, the equation

$$(1-4\sqrt{2}r+12r^2-10\sqrt{2}r^3+12r^4-4\sqrt{2}r^5+r^6)^2$$
$$+(1-2\sqrt{2}e^{\frac{1}{4}\pi i}r^3-r^6)^2=1+r^{12},$$

and $(\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'})\sqrt{2}r=-(\kappa\lambda+\kappa'\lambda')$

$$=-(1-4\sqrt{2}r+12r^2-10\sqrt{2}r^3+12r^4-4\sqrt{2}r^5+r^6).$$

This equation of the 12th degree contains factors corresponding to

$$\omega=2+\sqrt{-157},\ -6+\sqrt{-125},\ \text{and}\ 10+\sqrt{-61}.$$

23. *Moduli corresponding to*

$$\omega=\sqrt{-67},\ \sqrt{-59},\ \sqrt{-43},\ \sqrt{-19}.$$

If in the modular equation of the 17th order, viz.,

$$(v-u)^6-10\sqrt[3]{2}(v-u)^3(vuv'u')^4(v'u')^2-4\sqrt[3]{4}(vuv'u')^4(v'u')^5$$
$$-8\sqrt[3]{2}(vuv'u')^3v'u'=0,$$

where $u = \phi(\omega)$, $v = \phi(\Omega)$, and $\Omega = \frac{\omega+16s}{n}$,

we suppose $$\omega = 2r - \frac{1}{\Omega+1},$$

we easily find that

$$\omega = -\frac{(16s+n-2r)+\sqrt{-\left[4n-(16s+n+2r)^2\right]}}{2}.$$

Taking $s=-1$, $r=1$, $n=17$, we have

$$\omega = \frac{1+\sqrt{-59}}{2} \text{ and } \Omega+1 = -\frac{1}{\omega-2};$$

hence $$e^{\frac{1}{24}\pi i}\frac{\phi(\Omega)}{\psi(\Omega)} = \psi(\omega), \quad \frac{1}{\psi(\Omega)} = e^{-\frac{1}{24}\pi i}\phi(\omega),$$

or $$v = e^{\frac{1}{12}\pi i}\frac{\omega}{u}, \quad v' = \frac{e^{\frac{1}{12}\pi i}}{u}.$$

If now we assume $\frac{v}{u} = t^3$, we have immediately

$$u' = e^{-\frac{1}{12}\pi i}u^3t^3, \quad v' = \frac{e^{\frac{1}{12}\pi i}}{u}, \quad u'v' = e^{\frac{1}{12}\pi i}ut^3;$$

whence, substituting in the modular equation, we obtain the equation

$$(1-t^3)^6+10\sqrt[3]{2}\,(1-t^3)^2t^{10}e^{\frac{1}{3}(4\pi i)}-4\sqrt[3]{4}\,t^{17}e^{\frac{1}{3}(2\pi i)}+8\sqrt[3]{2}\,t^{18}e^{\frac{1}{3}(4\pi i)} = 0.$$

Let us examine what t really represents.

$$t^3 = \frac{u}{v} = e^{\frac{1}{12}\pi i}\frac{u'}{u^2} = e^{\frac{1}{12}\pi i}\,\frac{\psi\left(\frac{\sigma+1}{2}\right)}{\phi^2\left(\frac{\sigma+1}{2}\right)} \text{ where } \sigma = \sqrt{-59};$$

therefore $$t^3 = e^{\frac{1}{12}\pi i}\frac{\psi^2(\sigma+1)}{\sqrt{2}\,\phi(\sigma+1)}$$

$$\left(\text{modular equation of second order is } \frac{\psi(\omega)}{\phi^2(\omega)} = \frac{1}{\sqrt{2}}\frac{\psi^2(2\omega)}{\phi(2\omega)}\right),$$

or $$t^3 = \frac{1}{\sqrt{2}\,\phi(\sigma)\,\psi(\sigma)} = \frac{1}{\sqrt[4]{4\kappa\kappa'}}.$$

We have, therefore, for $r = \frac{1}{\sqrt[12]{4\kappa\kappa'}}$, the equation

$$(1-r^3)^6+10\sqrt{2}\,(1-r^3)^2r^{10}-4\sqrt[4]{4}\,r^{17}+8\sqrt[3]{2}\,r^{18} = 0,$$

which breaks up into the factors

$$r^3-\sqrt[3]{2}\,r-1,\quad r^3-\sqrt[3]{4}\,r^3-1,\quad r^3-\sqrt[3]{4}\,r^2-\sqrt[3]{2}\,r-1,$$

and $r^9-2\sqrt[3]{4}\,r^8+2\sqrt[3]{2}\,r^7-r^6+\sqrt[3]{4}\,r^5-2\sqrt[3]{2}\,r^4+r^3-\sqrt{4}\,r^2+2\sqrt[3]{2}\,r-1,$

corresponding, respectively, to

$$\omega=\sqrt{-19},\quad \sqrt{-43},\quad \sqrt{-67},\quad \sqrt{-59}.$$

(See Greenhill, *Proceedings*, Vol. XIX., p. 311.)

The Equation of 3rd and 5th Orders.

24. The equation of the 3rd order is

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1,$$

whence it immediately follows that

$$\kappa\lambda+\kappa'\lambda'=\sqrt{4\kappa\lambda\kappa'\lambda'},$$

or $$(\kappa\lambda'+\kappa'\lambda)^2=1+4\kappa\lambda\kappa'\lambda'-(1-\sqrt{4\kappa\lambda\kappa'\lambda'})^2=2\sqrt{4\kappa\lambda\kappa'\lambda'},$$

and therefore the equation of the 3rd order may be written

$$\kappa\lambda'+\kappa'\lambda=\sqrt{2}\,\sqrt[4]{4\kappa\lambda\kappa'\lambda'}.$$

The equation of the 5th order ($n+1\equiv 0$, mod. 3) is remarkably simple when expressed in terms of $\kappa\lambda'$, $\kappa'\lambda$. The usual form in which it is written is

$$\kappa\lambda+\kappa'\lambda'+2\sqrt[3]{4\kappa\lambda\kappa'\lambda'}=1;$$

therefore $$(\kappa\lambda'+\kappa'\lambda)^2=t^{12}-4t^8+4t^4,\quad t=\sqrt[12]{4\kappa\lambda\kappa'\lambda'},$$
$$=(2t^3-t^6)^2;$$

hence $$\kappa\lambda'+\kappa'\lambda=2t^3-2\sqrt{\kappa\lambda\kappa'\lambda'};$$

and therefore equation of the 5th order is

$$\sqrt{\kappa\lambda'}+\sqrt{\kappa'\lambda}=\sqrt{2}\,\sqrt[12]{4\kappa\lambda\kappa'\lambda'}.$$

If we combine the equations of 3rd and 47th orders according to the general theory, then, in order that

$$mnt-8r-8ms=2,\quad m=47,\quad n=3,$$

we must have $t=2$, and then

$$mnt-8r-8ms=2,\ -6,\text{ or }10;$$

and therefore $\omega=2+\sqrt{-137},\ -6+\sqrt{-105},\ 10+\sqrt{-41},$

and $$\Omega=4-\frac{1}{\omega};$$

therefore the modular equations are to be written

$$P^3-4Q-(4R)^{\frac{1}{2}}P-2(4R)^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\lambda\kappa+\lambda'\kappa'=\sqrt{2}\,e^{\frac{1}{12}\pi i}\sqrt[4]{4\kappa\lambda\kappa'\lambda'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Rationalising (1) and replacing $\lambda\kappa+\lambda'\kappa'$ by $\sqrt{2}\,e^{\frac{1}{12}\pi i}t^3$, where

$$t=\sqrt[12]{4\kappa\lambda\kappa'\lambda'},$$

we have an equation in t of 12th degree corresponding to the values of ω given above.

25. The discussion of the equations of the 3rd and 23rd orders leads to the values of moduli corresponding to

$$\omega=\sqrt{-69},\quad \sqrt{-65},\quad \sqrt{-53},\quad \sqrt{-33},\quad \sqrt{-5}.$$

If $t=2$, then, taking $m=23$, $n=3$, we have

$$mnt-8r-8ms=138-8r-184=2 \text{ or } -6;$$

hence we may expect $2+\sqrt{-65}$ and $-6+\sqrt{-33}$ to make their appearance simultaneously.

The modular equations are to be written

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt{2}\sqrt[12]{4\kappa\lambda\kappa'\lambda'}=1,$$

$$\kappa\lambda+\kappa'\lambda'=\sqrt{2}\,e^{\frac{1}{12}\pi i}\sqrt[4]{4\kappa\lambda\kappa'\lambda'} \text{ or } -\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1.$$

Squaring the first twice, we have

$$\kappa\lambda+\kappa'\lambda'=1-4\sqrt{2}\,t+12t^2-10\sqrt{2}\,t^3+12t^4-4\sqrt{2}\,t^5+t^6;$$

hence the equation for t is

$$1-4\sqrt{2}\,t+12t^2-10\sqrt{2}\,t^3+12t^4-4\sqrt{2}\,t^5+t^6=t^3\sqrt{-2}.$$

Putting $$t+\frac{1}{t}=\frac{\psi}{\sqrt{2}},$$

this equation reduces to

$$[\psi-(3+\sqrt{-1})][\psi^2-(5-\sqrt{-1})\psi+(2-2\sqrt{-1})]=0,$$

of which the factors correspond to $-6+\sqrt{-33}$ and $2+\sqrt{-65}$, respectively.

From the equation $$-\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1$$

we deduce, by squaring, $\kappa'\lambda+\kappa\lambda'=1+t^6$

and $$\kappa\lambda+\kappa'\lambda'=\sqrt{-2}\,t^3;$$

therefore $$2\kappa\kappa'+2\lambda\lambda'=\sqrt{-2}\,t^3(1+t^6),$$

and $$\frac{1}{2\kappa\kappa'}+\frac{1}{2\lambda\lambda'}=\sqrt{-2}\,\frac{1+t^6}{t^9};$$

therefore $$2\kappa\kappa'+\frac{1}{2\kappa\kappa'}+2\lambda\lambda'+\frac{1}{2\lambda\lambda'}=\sqrt{-2}\left(t^3+\frac{1}{t^3}\right)\left(t^6+\frac{1}{t^6}\right);$$

and therefore, exactly as in the case of $\sqrt{-41}$, we shall get an equation of the 4th degree for $\frac{1}{2\kappa\kappa'}+2\kappa\kappa'$.*

The case of $$\omega=\sqrt{-33}$$

need not be separately discussed, as the results are well known (see Prof. Greenhill's paper).

26. *Moduli corresponding to* $\omega=\sqrt{-53}$.

Taking $t=4$, we have, since

$$\Omega=8-\frac{1}{\omega},$$

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt{2}\,\sqrt[12]{4\kappa\lambda\kappa'\lambda'}=1,$$

$$\kappa\lambda+\kappa'\lambda'=\sqrt{2}\,\sqrt[4]{4\kappa\lambda\kappa'\lambda'},\quad\text{or}\quad\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1;$$

hence, as in the last case,

$$1-4\sqrt{2}\,t+12t^2-10\sqrt{2}\,t^3+12t^4-4\sqrt{2}\,t^5+t^6=-\sqrt{2}\,t^3;$$

therefore $$t^6-4\sqrt{2}\,t^5+12t^4-9\sqrt{2}\,t^3+12t^2-4\sqrt{2}\,t+1=0;$$

or, putting $$t+\frac{1}{t}=\frac{\psi}{\sqrt{2}},$$

we have $$\psi^3-8\psi^2+18\psi-4=0,$$

also $$2\kappa\kappa'+2\lambda\lambda'=-\sqrt{2}\,t^3(1-t^6),$$

and $$\frac{1}{2\kappa\kappa'}+\frac{1}{2\lambda\lambda'}=-\sqrt{2}\,\frac{1-t^6}{t^9};$$

---

* The result for $\omega=\sqrt{-65}$ may be written $\frac{1}{\sqrt[12]{2\kappa\kappa'}}-\sqrt[12]{2\kappa\kappa'}=\sqrt[4]{8+\sqrt{65}}$, and may be obtained by combining the modular equations for $n=5$ and $n=13$. (See Greenhill, *Proceedings*, Vol. XIX., p. 332.)

therefore $\frac{1}{2\kappa\kappa'}-2\kappa\kappa'+\frac{1}{2\lambda\lambda'}-2\lambda\lambda'=\sqrt{2}\,\kappa\left[t^3+\frac{1}{t^3}-\left(t^2+\frac{1}{t^2}\right)\right]$

$$=-\sqrt{2}\left(t^3+\frac{1}{t^3}\right)\left(t^3-\frac{1}{t^3}\right)^2;$$

and, employing the same process as in the case of

$$\omega=\sqrt{-61},$$

we obtain a cubic equation for $\frac{1}{2\kappa\kappa'}-2\kappa\kappa'$.

As explained already, these equations are not the complete solutions, as we know that, in the case of

$$\omega=\sqrt{-65},$$

we obtain an equation for $\frac{1}{\sqrt[12]{2\kappa\kappa'}}-\sqrt[12]{2\kappa\kappa'}$; and in the case of

$$\omega=\sqrt{-53},$$

I have already given the equation for $\frac{1}{\sqrt[3]{2\kappa\kappa'}}-\sqrt[3]{2\kappa\kappa'}$.

[27. In Art. 15, it is shown that the Jacobian Modular Equations for $n\equiv 4p+1$ may be derived from those expressed in terms of P, Q, R, by putting

$$\rho P=(u^2-v^2)^2;\quad -4\rho^2 Q=8u^2v^2(u^4+v^4)+(u'v')^8;$$

$$\rho^3(2R)=(uvu'v')^2(u'v')^6;$$

or $$\rho P=\left(\frac{u^2-v^2}{uv}\right)^2;\quad -4\rho^2 Q=\frac{8(u^4+v^4)}{(uv)^2}+\left(\frac{u'^2v'^2}{uv}\right)^4;$$

$$\rho^3(2R)=\left(\frac{u'^2v'^2}{uv}\right)^4;$$

where $$\left(\frac{2}{\rho}\right)^3+\left(\frac{2}{\rho}\right)^2P+\left(\frac{2}{\rho}\right)Q+R=0,$$

or $$\frac{2}{\rho}=1,\quad -\kappa\lambda,\quad -\kappa'\lambda',$$

and it is there remarked that, if $F(u,v)=0$ be the Jacobian Equation, the result of the above substitution is

$$F(u,v).F(u,-v)=0.$$

Now, since $F(u, v) \cdot F(u, -v) \equiv (u^2-v^2)^{n+1}+\ldots$, *it is easy to see that* $F(u, v) = 0$ *takes either of the forms*

$$(u^2-v^2)^{\frac{1}{2}(n+1)}+\ldots = 0 \quad \textit{or} \quad (u-v)^{n+1}+\ldots = 0,$$

and these respectively correspond to

$$n = 4(2p+1)+1 \quad \textit{and} \quad n = 4(2p)+1.$$

For in the former case $(u-v)^{n+1}$ is obviously not rational in q.

28. In the transition from the P, Q, R to the Jacobian equations, it is important to notice that P and R are perfect squares; and, in particular, that P is the square of u^2-v^2.

We may, therefore, when $n = 8p+5$, *expect to find that the* P, Q, R *modular equations can be expressed as rational algebraic functions of* $\sqrt{P}$, Q, $\sqrt{R}$, *and when* $8p+5 = 0$, *mod.* 3, *in terms of the quantities* $\sqrt{P}$, Q, $\sqrt[6]{R}$. In other words, the equations for $n = 13$ and $n = 29$, which we have already determined, ought to reduce to the difference of two squares.

Such is the fact. We thus obtain,

for $$n = 13,$$

$$P^{\frac{1}{2}}(P^3+256R) = (32R)^{\frac{1}{2}}(11P^2-64Q) = 0,$$

and for $$n = 29,$$

$$P^{\frac{1}{2}}\left[P^2+17P(32R)^{\frac{1}{2}}-9(32R)^{\frac{2}{3}}\right]$$

$$-(32R)^{\frac{1}{2}}\left[9P^2-64Q-13P(32R)^{\frac{1}{2}}+15(32R)^{\frac{2}{3}}\right] = 0.$$

The existence of the cubic for ρ in the previous article arises from the fact that the Jacobian modular equations may be obtained from those in $\kappa\lambda$, $\kappa'\lambda'$ by any one of the three substitutions

$$(1)\ \kappa = \frac{2u^2}{1+u^4}, \quad \kappa' = \frac{1-u^4}{1+u^4}; \quad \lambda = \frac{2v^2}{1+v^4}, \quad \lambda' = \frac{1-v^4}{1+v^4}; \quad \rho = -\frac{2}{\kappa\lambda};$$

$$(2)\ \kappa = \frac{1-u^4}{1+u^4}, \quad \kappa' = \frac{2u^2}{1+u^4}; \quad \lambda = \frac{1-v^4}{1+v^4}, \quad \lambda' = \frac{2v^2}{1+v^4}; \quad \rho = -\frac{2}{\kappa'\lambda'};$$

$$(3)\ \kappa = \frac{1-u^4}{2\sqrt{-1}u^2}, \quad \kappa' = \frac{1+u^4}{2u^2}; \quad \lambda = \frac{1-v^4}{2\sqrt{-1}v^2}, \quad \lambda' = \frac{1+v^4}{2v^2}; \quad \rho = 2;$$

and, if the substitution in (3) be adopted, we may accurately

express P, Q, R in terms of u, v, as follows:—

$$\sqrt{2P} = \frac{u^3 - v^3}{uv} \text{ or } \sqrt{2(P+2)} - 2 = \frac{(u-v)^3}{uv},$$

$$-16Q = \frac{8(u^4+v^4)}{(uv)^2} + \left(\frac{u^2v^2}{uv}\right)^4,$$

$$16R = \left(\frac{u^2v^2}{uv}\right)^4;$$

and hence we infer that, if in the modular equation of the n^{th} order ($n \equiv 8p+1$) we eliminate Q by means of the identical relation

$$P + Q + R + 1 \equiv 0,$$

the resulting equation will break into two factors, each of which will be a rational algebraic function of $\sqrt{R}$, $\sqrt{P+2}$, and, when

$$8p+1 \equiv 0 \text{ (mod. 3)}, \text{ of } \sqrt[6]{R}, \sqrt{P+2}.$$

There will be no difficulty in verifying that the equation of the 17th order (see Art. 5) may be written

$$4(32R) - 12(32R)^{\frac{1}{2}} + 12P + 32 \pm \sqrt{2(P+2)}\left[2(32R)^{\frac{1}{2}} - 2P - 16\right] = 0.$$

29. *Modular Equation for* $n = 53$. The five coefficients which we have already determined in Art. 12, enable us to write the equation in the form

$$P^{\frac{1}{2}}\left[P^4 + (32R)^{\frac{1}{2}}(413P^3 - 2^{10}PQ) + \dots\right]$$

$$-(32R)^{\frac{1}{2}}\left[35P^4 - 2^6 \cdot 17QP^2 + 2^{12}Q^2 + (32R)^{\frac{1}{2}}(aP^3 + bPQ) + \dots\right] = 0;$$

and if in this we put

$$\sqrt{\lambda} = e^{\frac{1}{4}\pi}\sqrt{\kappa'}, \quad \sqrt{\lambda'} = \sqrt{\kappa},$$

or, in other words,

$$P = -1, \quad Q = -R,$$

we obtain factors in $(32R)^{\frac{1}{2}}$ corresponding to

$$\omega = 1 + \sqrt{-52}, \quad -3 + \sqrt{-44}, \quad 5 + \sqrt{-28}, \quad -7 + \sqrt{-4}.$$

Now, in these several cases, what does $(32R)^{\frac{1}{2}}$ actually represent? In the first case

$$(32R)^{\frac{1}{2}} = \sqrt[6]{32}\left\{\phi(1+\sqrt{-52})\,\psi(1+\sqrt{-52})\right\}^2$$

$$= -\sqrt{-1}\,\sqrt[6]{32}\left(\frac{\phi\sqrt{-52}}{\psi^2\sqrt{-52}}\right)^2$$

$$= -\frac{\sqrt{-1}}{x} \text{ suppose.}$$

In the other three cases, identically similar results are obtained. The equations in x, corresponding to the four values of ω, are,

for $\omega = 1+\sqrt{-52}$, $\quad x^4-16x^3+9x^2-2x-1=0$;

for $\omega = -3+\sqrt{-44}$, $\quad x^6-12x^5+8x^4-14x^3+12x^2-4x-2=0$;

for $\omega = 5+\sqrt{-28}$, $\quad x^2-6x+2=0$;

for $\omega = -7+\sqrt{-4}$, $\quad x-1=0$;

hence the product* of these four factors must be equal to the result of putting $\sqrt{P}=\sqrt{-1}$, $Q=-R$ and $\sqrt[6]{32R}=-\dfrac{\sqrt{-1}}{x}$ in the modular equation. Equating these results, we obtain the remaining coefficients, and arrive at the formidable result for $n=53$, viz.:

$$P^{\frac{1}{2}}\left[P^4+(32R)^{\frac{1}{3}}(413P^3-2^{10}PQ)+(32R)^{\frac{2}{3}}(3823P^2-2^6.101Q)\right.$$
$$\left.+2^8.703RP+2280\,(32R)^{\frac{5}{3}}\right]$$
$$=(32R)^{\frac{1}{2}}\left[35P^4-2^6.17P^2Q+2^{12}Q^2+(32R)^{\frac{1}{3}}(1941P^3-2^6.75PQ)\right.$$
$$\left.+(32R)^{\frac{2}{3}}(5077P^2-2^6.47Q)+4648\,.\,32RP+480\,(32R)^{\frac{4}{3}}\right].$$

30. *Modular Equation for $n=9$.* The equation in this case is

$$P^6+32R\,(64QP-14P^3)-3\,(32R)^2=0,$$

which breaks up into the factors

$$F(u,v)\,.\,F(u,-v),$$

where the first factor corresponds to the Jacobian modular equation, and may be written

$$(v'u')^8=\frac{(u-v)^8\,(u^2-v^2)^2}{16uv\,(u^2+v^2-uv)}.$$

In general, when $n=8p+1$, the Jacobian equation is the easier one to obtain directly.

31. *Modular Equation for $n=35$.* In this example, n being of the form $8p+3$, the modular equation will belong to Class III., and one-fourth the sum of the divisors of 35 will be its degree. We may

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* The product is

$$x^{13}-35x^{12}+413x^{11}-1941x^{10}+3823x^9-5077x^8+5624x^7-4614x^6$$
$$+2248x^5-330x^4-202x^3+94x^2+\,.\,-4.$$

therefore assume the equation to be of the form

$$P^4+(2R)^{\frac{1}{2}}(aP^3+bPQ)+(2R)^{\frac{1}{2}}(dP^2+eQ)+2fRP+g(2R)^{\frac{1}{2}}=0.$$

In this putting

$$\sqrt{\lambda}=\sqrt{\kappa'},\quad \sqrt{\lambda'}=\sqrt{\kappa},$$

we obtain an equation for

$$z=\sqrt[6]{4\kappa\kappa'}$$

corresponding to $\omega=\sqrt{-35},\quad 4+\sqrt{-19}.$

In fact, we obtain

$$(1-4\sqrt[3]{2}z+5\sqrt[3]{4}z^2-6z^3+2\sqrt[3]{2}z^4-\sqrt[3]{4}z^5+z^6)(1+2\sqrt[3]{2}z+\sqrt[3]{4}z^2+z^3)^2,$$

which at once leads to the required result

$$P^4+(2R)^{\frac{1}{2}}(-10P^3+32PQ)+8(2R)^{\frac{1}{2}}P^2-16RP-16(2R)^{\frac{1}{2}}=0.$$

The coefficients might have been more readily obtained by putting

$$\sqrt{\lambda}=-\sqrt{\kappa'},\quad \sqrt{\lambda'}=\sqrt{\kappa}.$$

We then derive an equation for

$$\omega=2+\sqrt{-31}.$$

This enables us to determine all the coefficients except one, and that one can be immediately found from the "q" formulæ or otherwise.

32. *The Modular Equation for* $n=95$ belongs to Class I., and its degree is one-eighth of the sum of the divisors of 95. We may therefore write it

$$(P^2-4Q)^2P+(4R)^{\frac{1}{2}}(aP^4+bQP^2+cQ^2)+\ldots\quad=0,$$

and if in this we put

$$\sqrt[4]{\lambda}=-\sqrt[4]{\kappa'},\quad \sqrt[4]{\lambda'}=\sqrt[4]{\kappa},\quad z=\sqrt[6]{16\kappa\kappa'},$$

we obtain a quintic in z for

$$\omega=4+\sqrt{-79}$$

(see Art. 7), which enables us to determine 5 of the coefficients.

Again, if we put $\sqrt[4]{\lambda}=\sqrt[4]{\kappa'}$, $\sqrt[4]{\lambda'}=\sqrt[4]{\kappa}$, we obtain, for $z=\sqrt[12]{16\kappa\kappa'}$,

$$(z^3+z+1)^2(z^8+\ldots\quad -1)=0,$$

corresponding to

$$\omega=8+\sqrt{-31},\quad \sqrt{-95}.$$

By comparing this product with the result of substitution in the modular equation, we derive the equation corresponding to

$$\omega = \sqrt{-95},$$

viz., $$z^8 - z^7 + \ldots + z^5 - 2z^4 - z^3 + 2z^2 + 2z - 1 = 0^*;$$

and the modular equation then becomes

$$(P^3 - 4Q)^3 P + (4R)^{\frac{1}{2}}(16QP^2 - 5P^4) + 4PQ(4R)^{\frac{1}{2}} - 4R(P^3 + 4Q) - 5P(4R)^{\frac{1}{2}} - 2(4R)^{\frac{1}{2}} = 0.$$

It would, perhaps, be easier to find a few coefficients by means of the "q" formulæ.

33. The last modular equation which we shall consider is that for $n = 119$. Precisely as in the case for $n = 95$, we recognise that its degree is 18, and therefore may be written

$$P^6 + (4R)^{\frac{1}{2}}(aP^5 + bP^3Q + cPQ^2) + (4R)^{\frac{2}{3}}(dP^4 + eP^2Q + fQ^2) + 4R(gP^3 + hPQ) + (4R)^{\frac{1}{2}}(iP^2 + jQ) + (4R)^{\frac{1}{2}}lP + m(4R)^3 = 0.$$

Dividing by $(4R)^{\frac{1}{2}}P^5$, and replacing

$$\frac{P}{(4R)^{\frac{1}{2}}} \text{ by } \frac{1}{q^4}(1 + q^2 + q^4 + q^6 + 2q^8 + 2q^{10} + 3q^{12} + \ldots),$$

$$\frac{(4R)^{\frac{1}{2}}}{P} \text{ by } q^4(1 - q^2 - q^5 + q^{10} - q^{12} \ldots),$$

$$\frac{-4Q}{P^2} \text{ by } \frac{1}{q^2}(1 - 3q^2 + 3q^4 - 2q^6 + q^8 + 2q^{10} - 4q^{12} + \ldots),$$

we easily find

$$c = -16, \quad b = 28, \quad a = -7, \quad e = 8, \quad d = -1, \quad h = -8, \quad g = 1.$$

Again, if we put

$$\sqrt{\lambda} = -\sqrt{\kappa'}, \quad \sqrt{\lambda'} = \sqrt{\kappa},$$

we obtain for $$\theta = -\sqrt[6]{16\kappa\kappa'}$$

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* Greenhill obtains

$$z = \frac{\sqrt{2\sqrt{5}-1}-1}{2} \cdot \frac{\sqrt{5}+1}{4}\left(-1 + 4\sqrt{\sqrt{5} + (\sqrt{5}-1)\sqrt{2\sqrt{5}-1}}\right)$$

as the positive root of this equation.

an equation corresponding to

$$\omega = \sqrt{-103}.$$

This equation is known to be of the 5th degree, and is of the form

$$\theta^5 + A\theta^4 + B\theta^3 + C\theta^2 + D\theta - 1 = 0.$$

Making the substitution, we see at once that

$$f = 0, \quad j = -\frac{c}{4}, \quad m = -\frac{h}{4}, \quad l + \frac{e}{4} = -1;$$

and therefore

$$j = 4, \quad m = 2, \quad l = -3;$$

i alone remaining unknown. We, however, obtain for the quintic corresponding to

$$\omega = \sqrt{-103},$$

$$z^{10} - (i+7)\, z^5 - z^6 + z^4 + 7z^3 - 1 = 0,$$

where

$$z = \sqrt[12]{16\kappa\kappa'},$$

which we know must reduce to the difference of two squares with integral coefficients. I leave the solution of this problem as an interesting example on indeterminate equations. The result is that $i = -9$, and we have, consequently, for

$$\omega = \sqrt{-103},$$

$$z^5 + 2z^4 + 3z^3 + 3z^2 + z - 1 = 0,$$

and the modular equation becomes

$$I^6 + (4R)^{\frac{1}{2}}(-7I^5 + 28I^3Q - 16PQ^2) + (4R)^{\frac{1}{2}} P\,(P^2 - 8Q)(\overline{4R}^{\frac{1}{2}} - P)$$
$$+ (4R)^{\frac{1}{2}}(4Q - 9P^2) - 3P\,(4R)^{\frac{1}{2}} + 2\,(4R)^2 = 0.$$

The following is a list of the modular equations with which we have been dealing :—

CLASS I.

$n = 15$, mod. 16.

$$I = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + 1, \quad Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'} + \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'}, \quad R = \sqrt[4]{\kappa\lambda\kappa'\lambda'}.$$

$n = 15, \quad P\,(I^2 - 4Q) + 4R = 0.$

$n = 31, \quad (I^2 - 4Q)^2 - 4IR = 0.$

$n = 47, \quad I^2 - 4Q - P\,(4R)^{\frac{1}{2}} - 2\,(4R)^{\frac{1}{2}} = 0.$

$n = 95$, $(P^2-4Q)^2 P+(4R)^{\frac{1}{2}}(16QP^2-5P^4)$
$+4PQ(4R)^{\frac{1}{2}}-4R(P^2+4Q)-5P(4R)^{\frac{1}{2}}-2(4R)^{\frac{1}{2}}=0.$

$n = 143$, $(P^2-4Q)^3 P+\ldots = 0.$

Class II.

$n \equiv 7$, mod. 16.

$P=\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1$, $Q=\sqrt[4]{\kappa\lambda\kappa'\lambda'}-\sqrt[4]{\kappa\lambda}-\sqrt[4]{\kappa'\lambda'}$, $R=-\sqrt[4]{\kappa\lambda\kappa'\lambda'}$.

$n = 7$, $P = 0$, or $(\sqrt[3]{2\kappa\kappa'}+\sqrt[3]{2\lambda\lambda'})\sqrt[12]{4\kappa\kappa'\lambda\lambda'}=\kappa\lambda'+\kappa'\lambda$.

$n = 23$, $P-(4R)^{\frac{1}{2}}=0$, $\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt{2}\sqrt[12]{4\kappa\lambda\kappa'\lambda'}=1$.

$n = 71$, $P^3+4(4R)^{\frac{1}{2}}(Q-P^2)+2P(4R)^{\frac{1}{2}}-(4R)=0$.

$n = 119$, $P^3+(4R)^{\frac{1}{2}}(-7P^5+28P^3Q-16PQ^2)$
$+(4R)^{\frac{1}{2}}P(P^2-8Q)(4R^{\frac{1}{2}}-P)$
$+(4R)^{\frac{1}{2}}(4Q-9P^2)-3P(4R)^{\frac{1}{2}}+2(4R)^2=0.$

Class III.

$n \equiv 3$, mod. 8.

$P=\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1$, $Q=\sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'}$, $R=-\sqrt{\kappa\lambda\kappa'\lambda'}$.

$n = 3$, $P = 0$.

$n = 11$, $P-(16R)^{\frac{1}{2}}=0$, $\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{4\kappa\lambda\kappa'\lambda'}=1$.

$n = 19$, $P^5-112P^3R+256QR=0$.

$n = 35$, $P^4+(16R)^{\frac{1}{2}}(16PQ-5P^3)+2P^2(16R)^{\frac{1}{2}}-16RP-(16R)^{\frac{1}{2}}=0$.

$n = 43$, $P^{11}+16R(-4^3Q^4+4^6\cdot 19Q^3P^3-4^4\cdot 125Q^2P^6$
$+4^2\cdot 330QP^6-286P^8)+\ldots$

$n = 59$, $P^5-(16R)^{\frac{1}{2}}(14P^4-144P^2Q+256Q^2)+(16R)^{\frac{1}{2}}P(19P^2-16Q)$
$-16R(42P^2+96Q)+4P(16R)^{\frac{1}{2}}-8(16R)^{\frac{1}{2}}.$

$n = 83$, $P^7+\ldots$,

Class $\overline{24}$

$n = 1 \mod 4$ $n = 1 \mod 3$ $n = 3 \mod 3$.

$P = \alpha_1 - \alpha_1' - 1$ $Q = \alpha_1\alpha_1' - \alpha_1 - \alpha_1'$. $E = -\alpha_1\alpha_1'$.

$n = 1$. $P = 1$.

$n = 3$. $P^3 - 3P - 1 + 4E^3 - 4E^2 - 4E = 0$. $4E^3 P^3 = 1$.

or $\alpha_1^3 - \alpha_1'^3 - \dfrac{\alpha_1 - \alpha_1' + \alpha_1^2 - \alpha_1'^2}{\alpha_1\alpha_1' + \alpha_1^2 - \alpha_1'^2 - \alpha_1\alpha_1'} = 1$

$n = 17$. $P^9 - 4P^7 - 6P^6 + 8P^5 + 12P^4 - 4P^3 - 8P^2 + P - 32E = 0$.

or $\alpha_1^{\frac{1}{2}} - \alpha_1'^{\frac{1}{2}} + \alpha_1^{\frac{1}{4}} - \alpha_1'^{\frac{1}{4}} - (\alpha_1\alpha_1')^{\frac{1}{4}}(\alpha_1^{\frac{1}{2}} - \alpha_1'^{\frac{1}{2}}) + 4\sqrt{2}(\alpha_1^{\frac{1}{4}} - \alpha_1'^{\frac{1}{4}}) = 0$.

$$\alpha = \frac{\alpha_1}{\alpha_1'}$$

or $\dfrac{\alpha - \alpha^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} + 1} - \dfrac{2\alpha^{\frac{1}{2}}(\alpha^{\frac{1}{2}} - 1)}{\alpha + 1}\sqrt{\dfrac{\alpha - \alpha^{\frac{1}{2}} + 1}{\alpha}} - \left(\dfrac{2\alpha^{\frac{1}{2}}(\alpha^{\frac{1}{2}} - 1)}{\alpha + 1}\right)^2 = 0$.

$n = 5$. $P - (32E)^2 = 1$. $\alpha_1 - \alpha_1' + 2\sqrt{4\alpha_1\alpha_1'} = 1$.

or $\dfrac{\alpha^2 - \alpha}{\alpha + 1} - \left(\dfrac{2\alpha^{\frac{1}{2}}(\alpha^{\frac{1}{2}} - 1)}{\alpha + 1}\right)^2 = 0$.

$n = 13$. $P^7 - P^6 - 256E^2 \pm (32E)^2(11P^2 - 64Q) = 0$.

$n = 29$, $P^{15} - P^{14} - 17P(32E)^2 - 9(32E)^3$

$$\pm (32E)^4 [9P^6 - 64Q - 13P(32E)^2 + 15(32E)^4] = 0.$$

$n = 53$, $P^{27}[P^6 + (32E)^2(413P^6 - 2^{11}PQ)$

$$+ (32E)^4(3523P^6 - 2^6 \cdot 101Q) - 2^6 703RP + 2280(32E)^6]$$

$$\pm (32E)^6[35P^6 - 2^6 \cdot 17P^4Q + 2^{12}Q^3$$

$$+ (32E)^2(1941P^6 - 2^6 \cdot 75Q) + (32E)^4(5077P^6 - 2^6 \cdot 47Q)$$

$$+ 4648 \cdot 32EP + 480(32E)^6] = 0.$$

Complex Multiplication Moduli.

34. I conclude the present contribution with a short discussion on the determination of the moduli corresponding to

$$\omega = \sqrt{-n},$$

where $n = 3, \mod 8$.

We have seen that the quantity in terms of which these equations are expressed is $\sqrt[12]{4\kappa\kappa'}$, unless n be divisible by 3, in which case it is $\sqrt[4]{4\kappa\kappa'}$. If, after Greenhill, we denote by s the quantity

$$\sqrt[12]{\frac{\kappa\kappa'}{4}},$$

we shall find that the required equations have integral coefficients.

The six simplest equations are as follows:—

$$2s^3-1=0, \qquad \omega=\sqrt{-3};$$
$$2s^3-2s^2+2s-1=0, \qquad \omega=\sqrt{-11};$$
$$2s^3+2s^2-1=0, \qquad \omega=\sqrt{-19};$$
$$2s^3+2s-1=0, \qquad \omega=\sqrt{-43};$$
$$2s^3+2s^2+2s-1=0, \qquad \omega=\sqrt{-67};$$
$$2s^3-4s^2+6s-1=0, \qquad \omega=\sqrt{-163};$$

and if for each we determine Hermite's "α" function, where

$$\alpha \equiv \frac{(1-256s^{24})^3}{256s^{24}} \equiv \frac{(1-16\kappa^2\kappa'^2)^3}{16\kappa^2\kappa'^2},$$

we shall in these several cases obtain

$$\frac{\alpha^{\frac{1}{3}}}{\sqrt[3]{2}}=0, \quad 2^2, \quad 2^2.3, \quad 2^2.3.5, \quad 2^2.3.5.11, \quad 2^2.3.5.23.29;$$

or, in other words, the cubic equations in s reduce to simple equations for

$$\sqrt[3]{\frac{\alpha}{2}}.$$

We shall similarly find in more difficult cases that the degree of the equation in α is one-third of the degree of the corresponding equation in s. For example,

$$8s^9-16s^8+8s^7-4s^6+8s^5-4s^4+2s^3-4s^2+4s-1=0,$$
$$\omega=\sqrt{-59},$$

gives

$$\left(\frac{\alpha}{2}\right)-392\left(\frac{\alpha}{2}\right)^{\frac{1}{3}}+1072\left(\frac{\alpha}{2}\right)^{\frac{2}{3}}-1408=0;$$

$$8s^9-8s^8.4s^6+8s^5-20s^4+22s^3-14s^2+6s-1=0,$$
$$\omega=\sqrt{-83},$$

gives $$\left(\frac{\alpha}{2}\right)-1740\left(\frac{\alpha}{2}\right)^{\frac{1}{3}}+2000\left(\frac{\alpha}{2}\right)^{\frac{1}{3}}-16000=0;$$

$$8s^9+16s^8+32s^7+36s^6+32s^5+24s^4+18s^3+8s^2-1=0,*$$

$$\omega=\sqrt{-107},$$

gives $$\left(\frac{\alpha}{2}\right)-79.80\left(\frac{\alpha}{2}\right)^{\frac{1}{3}}-69.800\left(\frac{\alpha}{2}\right)^{\frac{1}{3}}-17.8000=0.$$

From these results it seems clear that, if

$$2s^3-as^2+\beta s-1=0,$$

when rationalized, produces an equation of the form

$$256s^{24}+Bs^8-1=0,$$

then it is a proper modular function. Rationalizing, we obtain

$$256s^{24}-\left[\left\{(\beta^2-2\alpha)^2-2(\alpha^2-4\beta)\right\}\right.$$
$$\left.-2\left\{(\alpha^2-4\beta)^2-8(\beta^2-2\alpha)\right\}\right]s^8-1=0,$$

with the necessary condition

$$\left\{(\alpha^2-4\beta)^2-8(\beta^2-2\alpha)\right\}^2=32\left\{(\beta^2-2\alpha)^2-2(\alpha^2-4\beta)\right\}.$$

35. If in this equation of condition we suppose β known, we get eight values of α. As an example, let us take $\beta=2$.

The equation of the 8th degree now obviously contains as factors $\alpha-2$, α, $\alpha+2$, corresponding to $\omega=\sqrt{-11}$, $\sqrt{-43}$, $\sqrt{-67}$, respectively, and we are left with a quintic factor which breaks into the two $(\theta^2+\theta-3)(\theta^3-\theta^2-3\theta+4)$, where $\alpha=2\theta$.

The first is equivalent to

$$2s^3+(1+\sqrt{13})s^2+2s-1=0,$$

and corresponds to $\omega=\sqrt{-91}$.

The cubic factor with discriminant -83 corresponds to $\omega=\sqrt{-83}$.

The modulus equation, therefore, for $\omega=\sqrt{-83}$ is

$$\theta^3-\theta^2-3\theta+4=0, \quad \text{where} \quad \theta=\frac{2s^3+2s-1}{2s^2}.$$

---

* This result was obtained by treating the equation of the 29th order (see Art. 28) in the manner adopted in Art. 23. The factors for

$$\omega=\sqrt{-115}, \quad \sqrt{-91}, \quad \sqrt{-67}, \quad \sqrt{-35},$$

divided out, and the equation given above alone remained.

Professor Greenhill (*Proceedings*, Vol. XIX., p. 312), by employing Hermite's method to the modular equation of the 23rd order, obtained an equation of the 24th degree in s, which contains the factors corresponding to $\omega = \sqrt{-91}$, $\sqrt{-83}$, $\sqrt{-67}$, $\sqrt{-43}$, $\sqrt{-11}$. Recognising that the equations for $\omega = \sqrt{-11}$, $\sqrt{-43}$, $\sqrt{-67}$, $\sqrt{-91}$ were expressible in the form

$$\frac{1-2s-2s^2-2s^3}{2s^2} = \beta,$$

for different well-known values of β he assumed that the modulus equation for

$$\omega = \sqrt{-83}$$

was a cubic function of the same quantity. The solution is a very brilliant one, especially when we consider that it is absolutely the only case in which such a series of coincidences arises when a modular equation is treated by Hermite's method.

The roots of all equations such as the equation of the 9th degree for

$$\omega = \sqrt{-83}$$

can be expressed in radicals, and "Greenhill's auxiliary functions" supply us with the very simplest method for attaining that end. It was the consideration of the idea embodied in the above solution which led me to the present point of view.

36. As another example, let us take $\alpha = \beta$. Here we have an equation of the 8th degree in β, which obviously contains as factors β, $(\beta-2)$, $\{\beta-(1\pm\sqrt{5})\}$, corresponding respectively to

$$\omega = \sqrt{-3},\ \sqrt{-11},\ \sqrt{-35},\ \text{and}\ \sqrt{-\frac{7}{5}},$$

leaving a quartic factor which breaks into two quadratics, giving

$$\beta = 3 \pm \sqrt{5},$$

corresponding to $\omega = \sqrt{-115}$ and $\sqrt{-\frac{23}{5}}$;

and

$$\beta = 3 \pm \sqrt{17},$$

corresponding to $\omega = \sqrt{-187}$ and $\sqrt{-\frac{17}{11}}$

In this case we observe that "Greenhill's auxiliary function" is

$$\frac{2s^3-1}{s^3-s}=\beta.$$

37. Again, taking

$$a=-\beta \text{ and } \frac{1-2s^3}{2(s^3+s)}=\beta$$

for the auxiliary function, we have for β an equation of the 8th d which breaks into the factors

$$\beta(\beta-1)(\beta^3-3\beta^2+\beta-2)(\beta^3-4\beta^2-1)=0,*$$

corresponding, respectively, to

$$\omega=\sqrt{-3},\ \sqrt{-67},\ \sqrt{-211},\ \sqrt{-283}.$$

If

$$\frac{1+2s^3-2s^4}{2s}=\beta$$

be chosen as auxiliary function, we obtain

$$\beta-1=0,$$

corresponding to $\omega=\sqrt{-11}$;

$$\beta^3-7\beta^2+9\beta-4=0$$

(discriminant -331), corresponding to

$$\omega=\sqrt{-331}.$$

In a similar manner, we see that

$$2s^4+2s^3+(7\pm\sqrt{61})\,s-1=0$$

is the modulus equation for

$$\omega=\sqrt{-427} \text{ or } \sqrt{-\frac{61}{7}},$$

and

$$\beta^3-29\beta^2+85\beta-66=0,$$

where

$$\beta=\frac{1+4s^3-2s^4}{2s}$$

is the modulus equation for $\omega=\sqrt{-907}$.

It may also be easily proved that the equation for $\omega=\sqrt{-59}$ (34) may be written

$$2s^3+2\,\frac{1+\beta}{1-\beta}s^2+2\beta s-1=0, \text{ where } \beta^3-2\beta^2-1=0;$$

and the equation for $\omega=\sqrt{-107}$,

$$2s^3+2\,\frac{\beta-1}{\beta+1}s^2+2\beta s-1=0, \text{ where } \beta^3-\beta-4=0.$$

* Discriminants of these cubics are -211 and -283.

38. The modulus equation for

$$\omega = \sqrt{-75} = 5\sqrt{-3}$$

is very readily obtained by substituting for $\lambda\lambda'$ in the modular equation of the 5th order the value $\frac{1}{4}$. The resulting equation is

$$4s^6 + 4\sqrt[3]{2}\, s^5 - 2\sqrt[3]{2}\, s + 1 = 0,$$

which breaks into the quadratics

$$2s^3 + s^2\sqrt[3]{2}\,(1 \pm \sqrt{5}) + s\sqrt[3]{4} - 1 = 0.$$

It is remarkable that the solution of these cubics gives

$$\sqrt[3]{16\kappa^3\kappa'^3} = (2 \pm \sqrt{5})\left[\pm\frac{1 \pm 3\sqrt{5}}{2}\sqrt[6]{5} - 4\sqrt[3]{2}\right],$$

corresponding to $\omega = 5\sqrt{-3}$ and $\dfrac{\sqrt{-3}}{5}$.

The modulus equation in this case suggests $\dfrac{1 - s\sqrt[3]{4} - 2s^3}{s^2\sqrt[3]{2}}$ as auxiliary function, but I have made no attempt to examine what consequences follow from such an assumption.]

On the Function which denotes the excess of the Number of Divisors of a Number which $\equiv 1$, *mod.* 3, *over the number which* $\equiv 2$, *mod.* 3.

By J. W. L. Glaisher, Sc.D., F.R.S.

[*Read May 8th*, 1890.]

1. In a paper "On the Square of Euler's Series,"* use was made of the function $H(n)$ which expresses the excess of the number of divisors of n which $\equiv 1$, mod. 3, over the number of divisors which $\equiv 2$, mod. 3. In the present paper I propose to consider the properties of this function more fully than was possible in a paper in which it occupied only a subordinate place. It will be seen that the function

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* *Proc. Lond. Math. Soc.*, Vol. xxi., pp. 182–215. See especially pp. 198–201, 209–212.

$H(n)$ is very similar, as regards its properties, to the function $E(n)$ which expresses the excess of the number of divisors of n which $\equiv 1$, mod. 4, over the number of divisors which $\equiv 3$, mod. 4 (Vol. xv. pp. 104–122).

2. It is evident that, if $n = 3^\rho r$, where r is not $\equiv 0$, mod. 3, then

$$H(n) = H(r);$$

and that, if $n \equiv 2$, mod. 3, then

$$H(n) = 0;$$

for to every divisor $\equiv 1$, mod. 3, there must correspond a conjugate divisor $\equiv 2$, mod. 3.

Also, if $n = 3^\rho uv$, where all the prime factors of u are $\equiv 1$, mod. 3, and all the prime factors of v are $\equiv 2$, mod. 3, then

$$H(n) = 0,$$

unless v is a square number, in which case

$$H(n) = \nu(u),$$

where $\nu(u)$ denotes the number of divisors of u.

If a be a prime $\equiv 1$, mod. 3, then

$$H(a^\alpha) = \alpha + 1;$$

and if r be a prime $\equiv 2$, mod. 3, then

$$H(r^\rho) = 1 \text{ or } 0,$$

according as ρ is even or uneven.

In general, if $\quad n = 3^\rho a^\alpha b^\beta c^\gamma \dots r^\sigma s^\tau t^\upsilon \dots,$

where $\quad a, b, c, \dots$ are primes which $\equiv 1$, mod. 3,

$r, s, t, \dots$ „ „ $\equiv 2$, mod. 3;

then $H(n)$ is zero, unless $\rho, \sigma, \tau, \dots$ are all even, and, if they are all even, then

$$H(n) = (\alpha+1)(\beta+1)(\gamma+1)\dots.$$

Also, if $n = n_1 n_2 n_3 \dots$, where $n_1, n_2, n_3, \dots$ are any relatively prime numbers, then

$$H(n) = H(n_1)\, H(n_2)\, H(n_3) \dots;$$

These theorems were stated in § 29, p. 199, of the paper on the square of Euler's Series, and two methods of proof were there indicated. The corresponding properties of the function $E(n)$ were proved in the first two sections of the paper in Vol. xv., which has been already referred to.

3. From Jacobi's theorem

$$\Pi_1^\infty (1-q^n) \times \sin x \Pi_1^\infty (1-2q^n \cos 2x + q^{2n}) = \Sigma_0^\infty (-1)^n q^{\frac{1}{2}n(n+1)} \sin (2n+1)x,$$

we find, by logarithmic differentiation,

$$\cot x + \frac{4q \sin 2x}{1-2q\cos 2x+q^2} + \frac{4q^2 \sin 2x}{1-2q^2\cos 2x+q^4} + \frac{4q^3 \sin 2x}{1-2q^3\cos 2x+q^6} + \&c.$$

$$= \frac{\cos x - 3q\cos 3x + 5q^3\cos 5x - 7q^6\cos 7x + 9q^{10}\cos 9x - \&c.}{\sin x - q\sin 3x + q^3\sin 5x - q^6\sin 7x + q^{10}\sin 9x - \&c.}.$$

Putting $x = \frac{1}{3}\pi$, this equation gives

$$1 + \frac{6q}{1+q+q^2} + \frac{6q^2}{1+q^2+q^4} + \frac{6q^3}{1+q^3+q^6} + \&c.$$

$$= \frac{1+6q+5q^3-7q^6-18q^{10}-11q^{15}+13q^{21}+\&c.}{1-q^3-q^6+q^{15}+q^{21}-q^{36}-q^{45}+\&c.}.$$

The left-hand member

$$= 1 + 6\frac{q-q^2}{1-q^3} + 6\frac{q^2-q^4}{1-q^6} + 6\frac{q^3-q^6}{1-q^9} + \&c.$$

$$= 1 + 6H(1)q + 6H(2)q^2 + 6H(3)q^3 + 6H(4)q^4 + \&c.,$$

whence $\Sigma_1^\infty H(n) q^n = \dfrac{q+q^3-q^6-3q^{10}-2q^{15}+2q^{21}+5q^{28}+\&c.}{1-q^3-q^6+q^{15}+q^{21}-q^{36}-q^{45}+\&c.}$.

Multiplying up by the denominator, and observing that $H(n) = 0$, if $n \equiv 2$, mod. 3, we find

$$q - 3q^{10} + 5q^{28} - 7q^{55} + \&c. + q^3 - q^6 - 2q^{15} + 2q^{21} + 3q^{36} - 3q^{45} + \&c.$$

$$= (1 - q^3 - q^6 + q^{15} + q^{21} - \&c.)$$

$$\times \{H(1)q + H(3)q^3 + H(4)q^4 + H(6)q^6 + H(7)q^7 + \&c.\},$$

which break up into the two equalities

$$1-3q^1-5q^3-7q^6-9q^{10}-\&c.$$
$$=1-q^1-q^2-q^3-q^4-\&c.\ \{H(1)+H(4)\,q^2+H(7)\,q^4+\&c.\},$$
$$q^1-q^2-3q^5-3q^7-5q^{12}-5q^{15}-\&c.$$
$$=1-q^1-q^2-q^3-q^4-\&c.\ \{H(3)\,q^1+H(6)\,q^3+H(9)\,q^5+\&c.\},$$

giving the formulæ

$$H(1)-H(4)\,q-H(7)\,q^2-H(10)\,q^3-\&c.$$
$$=\frac{1-3q^1-5q^3-7q^6-9q^{10}-\&c.}{1-q-q^2-q^3-q^4-\&c.},$$
$$H(1)\,q-H(2)\,q^2-H(3)\,q^3-H(4)\,q^4+\&c.$$
$$=\frac{q-q^2-2q^5+2q^7+3q^{12}-3q^{15}+\&c.}{1-q-q^2-q^3-q^4-\&c.}.$$

4. By equating coefficients we obtain the following recurring formulæ:—

(i.) If $n\equiv 1$, mod. 3, then

$$H(n)-H(n-3)-H(n-6)+H(n-15)+H(n-21)-\&c.$$

$=0$ or $(-1)^r(2r-1)$, according as n is not of the form $\frac{1}{2}(3r-1)(3r-2)$, or is equal to $\frac{1}{2}(3r+1)(3r+2)$.

(ii.) For all values of n,

$$H(n)-H(n-1)-H(n-2)+H(n-5)+H(n-7)-\&c.$$

$=0$, if n is not of the form $\frac{1}{2}r(3r\pm1)$,

$=(-1)^{r-1}r$, if $n=\frac{1}{2}r(3r-1)$,

$=(-1)^r r$, if $n=\frac{1}{2}r(3r+1)$.

The numbers $\frac{1}{2}(3r+1)(3r+2)$ which occur in the first formula are the triangular numbers which are not divisible by 3.

The numbers $\frac{1}{2}r(3r\pm1)$ which occur in the second formula are the pentagonal numbers, *i.e.*, the thirds of the triangular numbers which are divisible by 3.

5. The final results in § 3 bear a close analogy to the following formulæ relating to the function $E(n)$, which were given in §§ 7 and 9 of the paper in Vol. xv. (pp. 109, 110):

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c. = \frac{1-3q^2+5q^6-7q^{12}+\&c.}{1-2q+2q^4-2q^9+\&c.},$$

$$E(1)\,q+E(2)\,q^2+E(3)\,q^3+E(4)\,q^4+\&c. = \frac{q-q^3-2q^6+2q^{10}-\&c.}{1-q-q^3+q^6+q^{10}-\&c.}.$$

6. The last formula may be written

$$1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c. = \frac{1-3q+5q^3-7q^6+9q^{10}-\&c.}{1+q+q^3+q^3+q^{10}+\&c.};$$

so that we have

$$1-3q+5q^3-7q^6+\&c. = (1+q+q^3+q^6+q^{10}+\&c.)$$
$$\times\{1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c.\},$$

$$1-3q^2+5q^6-7q^{12}+\&c. = (1-2q+2q^4-2q^9+\&c.)$$
$$\times\{E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c.\},$$

$$1-3q^3+5q^9-7q^{18}+\&c. = (1-q-q^3+q^5+q^7-\&c.)$$
$$\times\{H(1)+H(4)\,q+H(7)\,q^2+H(10)\,q^3+\&c.\}.$$

Expressing the left-hand members of these equations as products, we have also

$$1-4E(1)\,q+4E(2)\,q^2-4E(3)\,q^3+\&c. = \left\{\frac{(1-q)(1-q^2)(1-q^3)\ldots}{(1+q)(1+q^2)(1+q^3)\ldots}\right\}^3,$$

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c. = \{(1+q)(1+q^2)(1+q^3)\ldots\}^4$$
$$\times\{(1-q)(1-q^2)(1-q^3)\ldots\}^3,$$

$$H(1)+H(4)\,q+H(7)\,q^2+H(10)\,q^3+\&c.$$
$$= \frac{\{(1-q^2)(1-q^4)(1-q^6)\ldots\}^3}{(1-q)(1-q^2)(1-q^3)\ldots}.$$

7. It was shown in the paper on the square of Euler's Series (Vol. xxi., p. 198) that

$$\Sigma_0^\infty H(4n+1)\,q^{4n+1} = \Sigma_0^\infty q^{m^2}\times\Sigma_{-\infty}^{\infty} q^{12n^2},$$

$$\Sigma_0^\infty H(4n+3)\,q^{4n+3} = \Sigma_{-\infty}^{\infty} q^{4n^2}\times\Sigma_0^\infty q^{3m^2},$$

$$H(1)+H(25)\,q+H(49)\,q^2+\&c.$$
$$=(1+q+q^3+q^5+\&c.)(1+2q^3+2q^8+2q^{18}+\&c.),$$
$$H(13)+H(37)\,q+H(61)\,q^2+\&c.$$
$$=2\,(1+q+q^3+q^5+\&c.)(1+q^4+q^{12}+q^{24}+\&c.).$$

Since

$$1+q+q^2+q^5+q^7+\&c.$$
$$=(1+q)(1+q^2)(1-q^3)(1+q^4)(1+q^5)(1-q^6)\ldots,$$
$$1+q+q^3+q^6+q^{10}+\&c.$$
$$=(1+q)(1+q^2)(1+q^3)\ldots(1-q^2)(1-q^4)(1-q^6)\ldots$$
$$=(1+q)(1+q^3)(1-q^4)(1+q^5)(1+q^7)(1-q^8)\ldots,$$
$$1+2q+2q^4+2q^9+2q^{16}+\&c.$$
$$=\frac{(1+q)(1-q^2)(1+q^3)(1-q^4)\ldots}{(1-q)(1+q^2)(1-q^3)(1+q^4)\ldots}$$
$$=(1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1+q^5)^2(1-q^6)\ldots,$$

we may at once express the above five H-series as products of binomial factors.

9. The series in which the coefficient of the general term is $H(m)$, m denoting any uneven number, may be expressed in terms of elliptic functions by the formulæ

$$k\rho\,\mathrm{cd}\,\tfrac{1}{3}K = k\rho\,\mathrm{sn}\,\tfrac{2}{3}K = 2\sqrt{3}\,.\,\Sigma_1^\infty H(m)\,q^{\frac{1}{3}m},$$

$$k\rho\,\mathrm{cn}\,\tfrac{1}{3}K = kk'\rho\,\mathrm{sd}\,\tfrac{2}{3}K = 2\sqrt{3}\,.\,\Sigma_1^\infty (-1)^{\frac{1}{3}(m-1)}\,H(m)\,q^{\frac{1}{3}m};$$

and the series in which the coefficient of the general term is $H(n)$ may be expressed in terms of Zeta functions by the formula

$$-\rho\,\mathrm{zc}\,\tfrac{1}{3}K = \rho\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1+6\,\Sigma_1^\infty H(n)\,q^{2n}\right\},$$

where
$$\mathrm{zc}\,x = \frac{i\pi}{2K}+\mathrm{zn}\,(x+K+iK') = \mathrm{zn}\,x-\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\mathrm{zs}\,x = \frac{i\pi}{2K}+\mathrm{zn}\,(x+iK') = \mathrm{zn}\,x+\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

$\mathrm{zn}\,x$ being the same as Jacobi's Zeta function $Z(x)$.

(10) I give below a table of the values of $\Sigma(n)$ as far as $n = 100$. The values of $\Sigma'(n)$ are added for the sake of comparison.

n	$\Sigma(n)$	$\Sigma'(n)$	n	$\Sigma(n)$	$\Sigma'(n)$	n	$H(n)$	$E(n)$
1	1	1	35	1	0	69	0	0
2	0	1	36	1	1	70	0	0
3	1	0	37	2	2	71	0	0
4	1	1	38	0	1	72	0	1
5	0	2	39	2	1	73	2	2
6	0	1	40	1	2	74	0	2
7	2	2	41	3	2	75	1	0
8	0	1	42	0	0	76	2	0
9	1	1	43	2	0	77	0	0
10	0	2	44	0	0	78	0	0
11	0	2	45	0	2	79	2	0
12	1	0	46	0	0	80	0	2
13	2	2	47	0	0	81	1	1
14	0	0	48	1	0	82	0	2
15	0	0	49	3	1	83	0	0
16	1	1	50	0	3	84	2	0
17	0	2	51	0	0	85	0	0
18	0	1	52	2	2	86	0	0
19	2	0	53	0	2	87	0	0
20	0	2	54	0	0	88	0	0
21	2	0	55	0	0	89	0	0
22	0	0	56	0	0	90	0	2
23	0	0	57	2	0	91	2	0
24	0	0	58	0	2	92	0	0
25	1	3	59	0	0	93	0	0
26	0	2	60	0	0	94	0	0
27	1	0	61	2	2	95	0	0
28	2	0	62	0	0	96	0	0
29	0	2	63	2	0	97	2	0
30	0	0	64	1	1	98	0	0
31	2	0	65	0	4	99	0	1
32	0	1	66	0	0	100	1	3
33	0	0	67	2	0			
34	0	2	68	0	2			

Table of Complex Multiplication Moduli. By A. G. GREENHILL.

[*Read May 8th*, 1890.]

A complete catalogue of all the numerical results in the solution of the modular equations for Complex Multiplication, when the square of the ratio of the periods is an integer, or a rational fraction, is of course impossible; but a list of those cases which have been solved up till now will be found useful as a verification of the form and coefficients of the modular equations, and also in some cases will serve for the determination of the coefficients when the form of the modular equation can be inferred from independent considerations.

It is proposed, then, to give here a list of these numerical results, as far as is known at present, embodying the values obtained by Abel, Jacobi, Kronecker, Hermite, Joubert, and Weber, and more recently of R. Russell and Mathews.

The numerical results are also presented in more than one shape, for verification in case of a misprint.

Denoting the ratio of the periods of the elliptic functions, K'/K, by $\sqrt{\Delta}$, and taking Δ as integral, it is convenient to draw up the list in four classes, according to the form of Δ (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 301); thus, in

$$\text{Class A,} \quad \Delta \equiv 3, \text{ mod. } 8;$$
$$\text{Class B,} \quad \Delta \equiv 7, \text{ mod. } 8;$$
$$\text{Class C,} \quad \Delta \equiv 1, \text{ mod. } 4;$$
$$\text{Class D,} \quad \Delta \equiv 2, \text{ mod. } 4;$$

the class for $\Delta \equiv 0$, mod. 4, not requiring much separate consideration.

Class A: $\Delta \equiv 3$, mod. 8.

Here the simplest numerical result to tabulate is Klein's *absolute invariant* J (*Math. Ann.*, Vol. XIV., p. 111), or Hermite's α (*Equations modulaires*), connected with J by the relation

$$J = -\frac{4}{27}\alpha;$$

[illegible] Legendre's modul[illegible] and [illegible] we may take

$$x = -\frac{[illegible]}{[illegible]}, \quad z = \frac{[illegible]}{[illegible]}$$

[illegible] [illegible] [illegible] and [illegible] that

$$x^2 - z = -[illegible]$$

so that

$$x_2 - [illegible] x_1 = -[illegible]$$

and γ_2 and γ_3 are then the [illegible] in the [illegible]

$\Delta = 3$; $J = [illegible]$. $\alpha = [illegible]$. $\sin\frac{2}{3}\omega = \frac{1}{2} = \sin\frac{1}{6}\pi$.

$\Delta = 11$; $J = -\frac{2^8}{11} \div \frac{2^8}{11}$. $\gamma_2 = \frac{1}{2}$. $\gamma_3 = 7 \cdot \sqrt{11} \div 2^7$.

$\Delta = 19$; $J = -\frac{2^8}{19}$. $\gamma_2 = 5$. $\gamma_3 = 9\sqrt{19}$. $\gamma_2 - 1 = 3^2$.

$$\gamma_3^2 - \gamma_2 - 1 = 3 \times 19.$$

$\Delta = 27$; $J = -2^8 \times 5^3 \div 3^9$. $\alpha = 2^2 \times 3 \times 5^3$,

$$\gamma_3 = -2^8 \times 5 \times 19.$$

$\Delta = 35$; $J = -\gamma_2^2$. $\gamma_2 = \frac{1}{4}\sqrt{5}\left\{\frac{1}{2}(\sqrt{5}+1)\right\}^6$,

$$\gamma_2 + 1 = \frac{1}{4}\left(\frac{\sqrt{5}+1}{2}\right)^6, \quad \gamma_2^2 - \gamma_2 + 1 = \frac{1}{3}(63+26\sqrt{5})^2,$$

$$\gamma_3 = (256+115\sqrt{5})\sqrt{7} \div 27.$$

$\Delta = 43$; $J = -2^{12} \times 5^3$. $\gamma_2 = 80$, $\gamma_3 = 21\sqrt{43}$,

$$\gamma_2 + 1 = 3^4, \quad \gamma_2 - \gamma_2 + 1 = 3 \times 7^2 \times 43.$$

$\Delta = 51$; $J = -64(5+\sqrt{17})^3(\sqrt{17}+4)^3$,

$$\gamma_3 = 7\sqrt{3}(128+31\sqrt{17}) \div 3^3 \text{ (Kiepert)}.$$

$\Delta = 59$; a, and therefore J, given by a cubic equation.
Mr. R. Russell finds that the equation for $z = \sqrt[12]{(4\kappa\kappa')}$ is

$$z^{24} - 2\tfrac{3}{2}z^{20} + \tfrac{3}{4}z^{17} - z^{16} + 2\tfrac{3}{2}z^{12} - \tfrac{3}{4}z^8 + z^4 - 2\tfrac{3}{2}z^3 + 2\tfrac{3}{4}z - 1 = 0;$$

and thence that the cubic equation for

$$\sqrt[3]{a} = (1-z^{24})/z^8$$

is $$a - 392\sqrt[3]{2}\,a^2 + 1072\sqrt[3]{4}\,a - 2816 = 0$$

and therefore $$27J + 7056J^2 + 12864J^3 + 2816 = 0.$$

$\Delta = 67$; $J = -2^6 \times 5^3 \times 11^3$, $\gamma_2 = 2^3 \times 5 \times 11$, $\gamma_3 = 217\sqrt{67}$,

$$\gamma_2 + 1 = 3^3 \times 7^2, \quad \gamma_2^2 - \gamma_2 + 1 = 3 \times 7^2 \times 31^2 \times 67.$$

$\Delta = 75$; $J = -64\sqrt{5}\,(31\sqrt{5}+69)$, $\gamma_2 = 4 \times 5^3\,(69+31\sqrt{5})$,

$$\gamma_3 = \tfrac{1}{9}\sqrt{3}\,(4352\sqrt{5}+9729).$$

Or, (Russell) with $z = \sqrt[12]{(4\kappa\kappa')}$,

$$z^6 + \sqrt[3]{4}\, z^5 - 2\sqrt[3]{2}\, z + 1 = 0.$$

$\Delta = 83$; a given by a cubic equation (Russell),

$$a - 1740\sqrt[3]{2}\, a^2 + 2000\sqrt[3]{4}\, a^2 - 32000 = 0.$$

$\Delta = 91 = 7 \times 13$; $\gamma_2 = 908 + 252\sqrt{13}$,

$$\gamma_3 = 11\sqrt{7}\,(2\sqrt{13}+7)(5\sqrt{13}+18), \quad \gamma_2 + 1 = 9\,(2\sqrt{13}+7)^2,$$

$$\gamma_2^2 - \gamma_2 + 1 = 3 \times 7 \times 11^2 \left\{\tfrac{1}{2}\,(\sqrt{13}+3)\right\}^6.$$

$\Delta = 99 = 3^2 \times 11$;

$$a = 2^6\,(4591804316 + 799330532\sqrt{33}) \quad \text{(Weber)}$$

$$= (\sqrt{33}+1)(55+\sqrt{33})^2\,(23+4\sqrt{33})^3.$$

$\Delta = 107$, a prime; Mr. Russell obtains a cubic for a,

$$a - 79 \times 80\sqrt[3]{2}\, a^2 - 69 \times 800\sqrt[3]{4}\, a^2 - 17 \times 16000 = 0.$$

$\Delta = 115 = 5 \times 23$;

$$J = -2^6 \times 5\sqrt{5}\,(157\sqrt{5}+351)^3, \quad \gamma_2 + 1 = 3^2\,(6\sqrt{5}+13)^2,$$

$$\gamma_3 = (6\sqrt{5}+13)(378+169\sqrt{5})\,\sqrt{23}.$$

$\Delta = 123 = 3 \times 41$; not yet solved.

$\Delta = 131$, and $\Delta = 139$, both primes; and requiring, according to Russell, quintic and cubic equations for a, respectively.

$\Delta = 147 = 3 \times 7^2$. Then (Russell), with $z = \sqrt[12]{(4\kappa\kappa')}$,

$$z^8 - 2z^7 + 7z^6 - 4z + 1 = 0.$$

$\Delta = 155 = 5 \times 31$; not yet solved.

$\Delta = 163$; $a = 2^{10} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$ (Hermite),

$$J = -\gamma_2^3, \quad \gamma_2 = 53360, \quad \gamma_3 = 185801\sqrt{163},$$

$$\gamma_2 + 1 = 3^3 \times 7^2 \times 11^2, \quad \gamma_2^2 - \gamma_2 + 1 = 3 \times 19^2 \times 127^2 \times 163.$$

Also, $2s^3-4s^2+6s-1=0$

gives $s=\sqrt[12]{(\frac{1}{4}\kappa\kappa')}$.

$\Delta=171=3^2\times19$; not yet solved.

$\Delta=179$; not yet solved.

$\Delta=187=11\times17$; $\beta=3+\sqrt{17}$,

where $\beta=\dfrac{1-2s^3}{s-s^3}$,

and $\sqrt[3]{2}\,s=\sqrt[12]{(4\kappa\kappa')}$ (Russell).

$\Delta=211$; $\beta^3-3\beta^2+\beta-2=0$,

where $\beta=\dfrac{1-2s^3}{2s+2s^2}$ (Russell).

$\Delta=283$; $\beta^3-4\beta^2-1=0$ (Russell).

$\Delta=331$; $\beta^3-7\beta^2+9\beta-4$, $\beta=\dfrac{1+2s^2-2s^3}{2s}$ (Russell).

$\Delta=427=7\times61$;

$2s^3+2s^2+(7+\sqrt{61})\,s-1=0$ (Russell).

$\Delta=907$; $\beta^3-29\beta^2+85\beta-66=0$, $\beta=\dfrac{1+4s^2-2s^3}{2s}$ (Russ

Class B. $\Delta\equiv7$, mod. 8.

Here the simplest function to tabulate seems to be $\kappa\kappa'$, or $\sqrt[4]{16\kappa\kappa'}$, and sometimes $\sqrt[12]{16\kappa\kappa'}$, when Δ is not a multiple of 3.

$\Delta=7$; $\sqrt[4]{\kappa\kappa'}=\frac{1}{2}$, $\sqrt[4]{16\kappa\kappa'}=1$, $\sqrt[12]{16\kappa\kappa'}=1$.

$\Delta=15=3\times5$; $\sqrt[4]{\kappa\kappa'}=\sin18°$, or $\sin54°=\frac{1}{4}(\sqrt5\pm1)$;

$\sqrt[4]{16\kappa\kappa'}=\frac{1}{2}(\sqrt5\pm1)$, $(16\kappa\kappa')^{-\frac{1}{4}}-(16\kappa\kappa')^{\frac{1}{4}}=1$.

$\Delta=23$; $x=\sqrt[12]{16\kappa\kappa'}$, given by the cubic equation x^3+x^2-1

$\Delta=31$; $x=\sqrt[12]{16\kappa\kappa'}$, given by the cubic equation x^3+x-1

$\Delta = 39$; $x = \sqrt[4]{16\kappa\kappa'}$, given by the equation

$$x^4+2x^3+4x^2+3x-1 = 0,$$

or

$$2x = -1+\{\tfrac{1}{2}(\sqrt{13}-1)\}^{\frac{1}{2}}$$

(Joubert, *Comptes Rendus*, t. 50).

$\Delta = 47$; $y = \sqrt[6]{16\kappa\kappa'}$ is given by the quintic equation

$$y^5+3y^2+2y-1 = 0;$$

and $x = \sqrt[12]{16\kappa\kappa'}$ by $x^5-2x^4+2x^3-x^2+1 = 0.$

This quintic is soluble by radicals; the real root, according to Prof. Paxton Young, being $u_1+u_2+u_3+u_4$, where, as simplified by Cayley (*American Journal of Maths.*, Vol. XIII., p. 53),

$$10^5u_1 = 2600\,(15+7\sqrt{5})+40\,(79+33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5+\sqrt{5})\}}\sqrt{47},$$

$$10^5u_2 = 2600\,(15-7\sqrt{5})-40\,(79-33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5-\sqrt{5})\}}\sqrt{47},$$

$$10^5u_3 = 2600\,(15-7\sqrt{5})+40\,(79-33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5-\sqrt{5})\}}\sqrt{47},$$

$$10^5u_4 = 2600\,(15+7\sqrt{5})-40\,(79+33\sqrt{5})\sqrt{\{\tfrac{1}{2}(5+\sqrt{5})\}}\sqrt{47}.$$

$\Delta = 55 = 5\times 11$; then, for

$$\mathrm{K}'/\mathrm{K} = \sqrt{55}, \quad \Lambda'/\Lambda = \sqrt{(11\div 5)},$$

$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \frac{7-\sqrt{5}}{8},$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = \frac{\sqrt{5}\sqrt{(10\sqrt{5}-18)}}{8},$$

obtained by the combination of the modular equations of the 5th and 11th orders (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 322), and then

$$\sqrt[4]{16\kappa\kappa'} = \{\sqrt{(6\sqrt{5}-2)}-(5-\sqrt{5})\}\div 4,$$

$$\sqrt[4]{16\lambda\lambda'} = \{\sqrt{(6\sqrt{5}-2)}+(5-\sqrt{5})\}\div 4;$$

and then (Russell)

$$\sqrt[12]{16\kappa\kappa'} = \{\sqrt{(6\sqrt{5}-2)}-\sqrt{5}+1\}\div 4,$$

$$\sqrt[12]{16\lambda\lambda'} = \{\sqrt{(6\sqrt{5}-2)}+\sqrt{5}-1\}\div 4;$$

so that $\sqrt[12]{16\kappa\kappa'}$ and $-\sqrt[12]{16\lambda\lambda'}$ are the real roots of the biquadratic equation

$$z^4-z^3+2z-1 = 0.$$

These results were obtained by Mr. Russell by putting

$$z' = -\tfrac{1}{2}z, \quad z' = \tfrac{1}{2}z$$

in the modular equation of the 7th order

$$\Delta = 13 = 13 \times 1^2 \quad z = \sqrt[24]{16\kappa\kappa'}$$ given by

$$z^2 - z - 1 = z - 1 = 1$$

(Proc. Camb. Phil. Soc. vol.).

$\Delta = 17$. Employing Russell's modular equation

$$P^3 - 4 - P^2 - Q^2 + 4Q^2 z - 2Q^2 + 4Q - 4Q = 0,$$

where $P = \tfrac{1}{2}\sqrt{\alpha} - \tfrac{1}{2}\sqrt{\alpha'} - 1$, $Q = \tfrac{1}{2}\sqrt{\alpha\alpha'} - \tfrac{1}{2}\sqrt{\alpha} - \tfrac{1}{2}\sqrt{\alpha'}$,

$$z = -\tfrac{1}{2}\sqrt{\alpha\alpha'};$$

and putting $\alpha' = \lambda$, $\alpha = \lambda'$, and $z = \sqrt[24]{16\kappa\kappa'}$, then

$$(z-1)^2(z^8 - z^7 - z^6 - z^5 - z^4 - z^3 - z^2 + 2z - 1) = 0,$$

the equation of the 7th degree being solvable by radicals, according to Abel.

$\Delta = 79$. The equation for $z = \sqrt[24]{16\kappa\kappa'}$ is, from Fiedler's modular equation,

$$(z^3 - z - 1)^3(z^5 - 4z^4 - 4z^3 - 5z^2 + 12z - 1) = 0,$$

the quintic equation being solvable by radicals. This is derived from Fiedler's modular equation for $n = 79$.

Then, according to Russell, the quintic factor reduces to

$$z^5 - z^4 + z^3 - 2z^2 + 3z - 1 = 0,$$

if

$$z = z^3 = \sqrt[8]{16\kappa\kappa'}.$$

$\Delta = 87 = 3 \times 29$; not yet solved.

$\Delta = 95 = 5 \times 19$. (G. B. Mathews.) With $y^{12} = 256\,\kappa\lambda\kappa'\lambda'$,

$$y^{12} - 7y^8 - 2y^7 - 7y^6 - (y-1)^5 = 0,$$

or $(y^8 + y^2 + 1)(y^9 - y^8 + y^7 - 2y^6 - 4y^5 + y^4 - 6y^3 + 9y^2 - 5y + 1) = 0,$

or $(y^8 + y^2 + 1)\{(2y^2 + 2y + 1)^2 - 5\}(y^5 - 3y^4 + 5y^3 - 7y^2 + 4y - 1) = 0;$

and then, according to Russell, the equation for

$$x = \sqrt[12]{16\kappa\kappa'},$$

is $$(2x^4-x^3+x^2+4x+1)^3 = 5\,(x^3+x^2+1)^3.$$

Then $$\left(2x^2-\frac{\sqrt5+1}{2}\,x+1\right)^3 = (2\sqrt5-1)\left(\frac{\sqrt5+1}{2}\,x-1\right)^3,$$

$$2x^2+\left\{\sqrt{(2\sqrt5-1)}-1\right\}\frac{\sqrt5+1}{2}\,x-\sqrt{(2\sqrt5-1)}+1=0,$$

$$\left[2x+\left\{\sqrt{(2\sqrt5-1)}-1\right\}\frac{\sqrt5+1}{4}\right]^2 = \frac{3\sqrt5-3+(5-\sqrt5)\sqrt{(2\sqrt5-1)}}{16},$$

whence $$x = \sqrt[12]{16\kappa\kappa'} \text{ or } \sqrt[12]{16\lambda\lambda'},$$

for $$\mathrm{K}'/\mathrm{K} = \sqrt{95}, \quad \Lambda'/\Lambda = \sqrt{(19\div5)}.$$

$\Delta = 103$; $111 = 3\times37$; $119 = 7\times17$; 127; $135 = 5\times27$; $143 = 11\times13$; ... not yet solved.

Class C: $\Delta \equiv 1$, mod. 4.

Here Hermite's $$\alpha = \frac{(1-4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2};$$

so that, if $$\beta = \tfrac12\sqrt{\alpha}, \quad \beta = (2\kappa\kappa')^{-1}-2\kappa\kappa';$$

and $\gamma = \sqrt{(\beta^2+4)} = (2\kappa\kappa')^{-1}+2\kappa\kappa'$, $\sqrt{(\gamma+2)} = (2\kappa\kappa')^{-\frac12}+(2\kappa\kappa')^{\frac12}$;

also denoting $(2\kappa\kappa')^{-\frac12}-(2\kappa\kappa')^{\frac12}$ by t,

$$t^3+3t = \beta.$$

$\Delta = 1$; $\beta = 0$, $2\kappa\kappa' = 1$, $J = 1$, $\gamma_3 = 0$.

$\Delta = 5$; $\beta = 4$, $(2\kappa\kappa')^{\frac12} = \frac12(\sqrt5-1)$, $t = 1$.

$\Delta = 9$; $\beta = 8\sqrt3$, $\gamma = 14$, $\sqrt{(\gamma+2)} = 4$,

$$2\kappa\kappa' = \left(\frac{\sqrt3-1}{\sqrt2}\right)^4;$$

$$(2\kappa\kappa')^{-\frac12}+(2\kappa\kappa')^{\frac12} = 4, \qquad (2\kappa\kappa')^{-\frac12}-(2\kappa\kappa')^{\frac12} = \sqrt2.$$

$\Delta = 13$; $\beta = 36$, $t = 3$, $(2\kappa\kappa')^{\frac12} = \frac12(\sqrt{13}-3)$.

$\Delta = 17$; $\gamma = 10\,(\sqrt{17}+4)$, $\sqrt{(\gamma+2)} = 5+\sqrt{17}$;

$$(2\kappa\kappa')^{-\frac12}+(2\kappa\kappa')^{\frac12} = \tfrac12(\sqrt{17}+1) \quad \text{(Weber)},$$

but immediately derivable from Mr. Russell's modular equation.

$$\Delta = 21; \quad \beta = 3(\sqrt{3}+1)^4, \quad \gamma = 32\sqrt{7}+18\sqrt{21};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2, \text{ for } \mathrm{K}'/\mathrm{K} = \sqrt{(21)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2, \text{ for } \Lambda'/\Lambda = \sqrt{(7\div 3)}.$$

The surds in the brackets are written so that the reciprocal is the complementary surd.

$$\Delta = 25 = 5^2; \quad \beta = 2^4\times 3^2\times\sqrt{5}, \quad \gamma = 322 = 2\times 7\times 23,$$

$$\sqrt{(\gamma+2)} = 18, \quad \sqrt[12]{2\kappa\kappa'} = \tfrac{1}{2}(\sqrt{5}-1);$$

so that $(2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}} = 1$, $(2\kappa\kappa')^{-\frac{1}{12}} + (2\kappa\kappa')^{\frac{1}{12}} = \sqrt{5}$.

$\Delta = 29$. Here β is given by the cubic equation

$$\beta^3 - 588\beta^2 - 976\beta - 3136 = 0,$$

and t by the cubic equation

$$t^3 - 9t^2 + 8t - 20 = 0;$$

so that $x = \sqrt[3]{2\kappa\kappa'}$ is given by the reciprocal sextic

$$x^6 + 9x^5 + 5x^4 + 2x^3 - 5x^2 + 9x - 1 = 0 \quad \text{(Weber)},$$

or by

$$2x^3 + 9x^2 + 8x + 5 = \sqrt{(29)}(x+1)^2.$$

This is immediately derivable from Russell's modular equation for $n = 29$ (*Proc. Lond. Math. Soc.*, Vol. XXI., a paper not yet published),

$$P^5 + (4R)^{\frac{1}{2}}(-94P^4 + 2^8\cdot 3^2 PQ - 2^{13}Q^2) + (4R)^{\frac{1}{2}}(2^2\cdot 505P^3 - 2^9\cdot 13PQ) + 4R(-2^3\cdot 745P^2 + 2^{10}\cdot 15PQ) + (4R)^{\frac{3}{2}}P - (4R)^{\frac{5}{2}} = 0;$$

or

$$P^{\frac{1}{2}}\{P^2 + 34P(4R)^{\frac{1}{2}} - 36(4R)^1\} \pm \sqrt{2}(4R)^{\frac{1}{2}}\{9P^2 - 64Q - 26P(4R)^{\frac{1}{2}} + 60(4R)^1\} = 0;$$

where

$$P = \kappa\lambda + \kappa'\lambda' - 1, \quad Q = \kappa\lambda\kappa'\lambda' - \kappa\lambda - \kappa'\lambda',$$

$$R = -\kappa\lambda\kappa'\lambda';$$

on putting $\kappa' = \lambda$, $\kappa = \lambda'$, and $x = \sqrt[3]{2\kappa\kappa'}$.

$\Delta = 33$; $\quad \alpha = 2^4 \times 3\,(75+13\sqrt{33})^3$,

$$\gamma = 10\,(52+1\sqrt{33}), \quad \sqrt{(\gamma+2)} = 3\,(\sqrt{33}+5),$$

$$2\kappa\kappa' = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6, \quad K'/K = \sqrt{33};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6, \quad \Lambda'/\Lambda = \sqrt{(11\div 3)}.$$

$\Delta = 37$; $\quad \alpha = 2^6 \times 3^4 \times 7^4$, $\quad \beta = 2^3 \times 3^3 \times 7^3$,

$$\gamma = 2 \times 5 \times 29 \times \sqrt{37},$$

$$2\kappa\kappa' = (\sqrt{37}-6)^3 \quad \text{(Kronecker)},$$

$$(2\kappa\kappa')^{-} - (2\kappa\kappa')^{\frac{1}{2}} = 12.$$

$\Delta = 41$. Weber's equation (*Acta Math.*, XI., p. 388) for

$$x = (2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}},$$

is $\quad x^3 - \frac{1}{2}\,(\sqrt{41}+5)^3 + \frac{1}{2}\,(7+\sqrt{41}) = 0,$

or $\quad x^4 - 5x^3 + 3x^2 + 3x + 2 = 0.$

$\Delta = 45 = 3^2 \times 5$. Here

$$\alpha = 2^6\,(17+10\sqrt{3})^4, \quad \beta = 2^3\,(17+10\sqrt{3})^3,$$

$$\gamma = 2\sqrt{5}\,(527+304\sqrt{3});$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^9 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4, \quad K'/K = \sqrt{(45)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^9 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4, \quad \Lambda'/\Lambda = \sqrt{(9\div 5)}.$$

$\Delta = 49 = 7^2$. Here

$$\alpha = 2^5 \times 3^4\,(3+\sqrt{7})^5\sqrt{7},$$

$$\gamma = 2\,(3^4 \times 23 + 2^6 \times 11\sqrt{7}), \quad \sqrt{(\gamma+2)} = 4\,(\sqrt{7}+2)^2;$$

and $\quad 2\kappa\kappa' = \left(\frac{\sqrt{7}+1-\sqrt{2}\sqrt[4]{7}}{2\sqrt{2}}\right)^{12},$

so that $\quad (2\kappa\kappa)^{\frac{1}{12}} + (2\kappa\kappa')^{-\frac{1}{12}} = \frac{\sqrt{7}+1}{\sqrt{2}}.$

$\Delta = 53$. According to Russell and Mathews, the equation for t is

$$t^3 - 23t^2 + 12t - 121 = 0,$$

or for $x = \sqrt[3]{2\kappa\kappa'}$ is the reciprocal sextic

$$x^6 + 23x^5 + 12x^4 + 75x^3 - 12x^2 + 23x - 1 = 0.$$

$\Delta = 57 = 3 \times 19$. Here

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^3, \quad \mathrm{K'/K} = \sqrt{57};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^3, \quad \Lambda'/\Lambda = \sqrt{(19 \div 3)}.$$

$\Delta = 61$. According to Mathews' result (*Proc. Lond. Math. Soc.*, Dec., 1889, Vol. XXI.), the equation for

$$t = (2\kappa\kappa')^{-\frac{1}{4}} - (2\kappa\kappa')^{\frac{1}{4}},$$

is $$t^3 - 30t^2 + 9t - 108 = 0,$$

or $$m^3 - 27m^2 - 5m - 1 = 0,$$

for $$a = 2^6 . 3^4 . m^4.$$

For $x = \sqrt[3]{2\kappa\kappa'}$, the equation

$$x^6 + 30x^5 + 6x^4 + 48x^3 - 6x^2 + 30x - 1 = 0.$$

$\Delta = 65 = 5 \times 13$. (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 332). On combining the modular equations of the 5th and 13th orders, and putting

$$4\kappa\lambda\,\kappa'\lambda' = x^6,$$

then $$x^{-1} + x = \tfrac{1}{2}(\sqrt{65} + 5), \quad x^{-\frac{1}{2}} + x^{\frac{1}{2}} = \tfrac{1}{2}(\sqrt{13} + \sqrt{5}),$$

and $$\kappa\kappa' + \lambda\lambda' = x^3 (2x^{-2} - 5 + 2x^2),$$

and then $$(2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}} = (2\lambda\lambda')^{-\frac{1}{12}} + (2\lambda\lambda')^{\frac{1}{12}}$$

$$= \sqrt[4]{(\sqrt{65} + 8)} \quad \text{(Russell)},$$

for $$\mathrm{K'/K} = \sqrt{(65)}, \quad \Lambda'/\Lambda = \sqrt{(13 \div 5)}.$$

$\Delta = 69 = 3 \times 23$; proceeding in a similar manner with

$$4\kappa\lambda\,\kappa'\lambda' = x^{12},$$

we find $x^{-1}+x = \frac{1}{2}(3\sqrt{2}+\sqrt{6})$, $\kappa\kappa'+\lambda\lambda' = \sqrt{2}\,x^3(1-x^6)$

(*Proc. Lond. Math. Soc.*, Vol. XIX., p. 332); and then (Russell) the equation for

$$\beta = (2\kappa\kappa')^{-1}-2\kappa\kappa',$$

when $K'/K = \sqrt{69}$, or $\sqrt{(23\div 3)}$; is

$$\beta^2-12\sqrt{3}(2+\sqrt{3})(374+216\sqrt{3})\beta+144(31+18\sqrt{3})^2 = 0.$$

$\Delta = 73$. We find, by approximate numerical calculation (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 363),

$$(2\kappa\kappa')^{-\frac{1}{2}}+(2\kappa\kappa')^{\frac{1}{2}} = \tfrac{1}{2}(\sqrt{73}+5),$$

so that $\gamma = 4930\sqrt{73}+42120$, $\sqrt{(\gamma+2)} = 17\sqrt{73}+145$.

$\Delta = 77 = 7\times 11$; (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 334) with

$$4\kappa\lambda\,\kappa'\lambda' = x^{12},$$

we find $x^{-1}+x = y = \frac{1}{2}(\sqrt{22}+\sqrt{2})$;

and $\kappa\kappa'+\lambda\lambda' = \sqrt{2}\,x(1-4x^3+4x^4-x^6)(2x-3x^3+2x^5)$.

Then (Russell) $z^3-(23+8\sqrt{11})z+4 = 0$

gives $z = (2\kappa\kappa')^{-\frac{1}{2}}-(2\kappa\kappa')^{\frac{1}{2}}$,

for $K'/K = \sqrt{(77)}$ or $\sqrt{(11\div 7)}$.

$\Delta = 81 = 3^4$; not yet solved.

$\Delta = 85 = 5\times 17$. Here

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{85}-9}{2}\right)^3, \quad K'/K = \sqrt{85};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^{12}\left(\frac{\sqrt{85}-9}{2}\right)^3, \quad \Lambda'/\Lambda = \sqrt{(17\div 5)};$$

so that $(2\kappa\kappa')^{-\frac{1}{2}}-(2\kappa\kappa')^{\frac{1}{2}} = \frac{3}{2}(21+5\sqrt{17})$,

$$(2\lambda\lambda')^{-\frac{1}{2}}-(2\lambda\lambda')^{\frac{1}{2}} = \tfrac{3}{2}(21-5\sqrt{17}).$$

$\Delta = 89$. Here $p = 6$ (Gauss); a 12-ic in $\sqrt[6]{2\kappa\kappa'}$ must be expected (Russell); not yet calculated.

$\Delta = 93 = 3 \times 31$;

$$2\kappa\kappa' = \left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^2 \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2, \quad K'/K = \sqrt{93};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^2 \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2, \quad \Lambda'/\Lambda = \sqrt{(31 \div 3)}.$$

$\Delta = 97$. By approximate calculation (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 364),

$$(2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}} = \tfrac{1}{2}(\sqrt{97}+9);$$

$$\sqrt{(\gamma+2)} = (2\kappa\kappa')^{-\frac{1}{4}} + (2\kappa\kappa')^{\frac{1}{4}} = 41\sqrt{97}+405.$$

$\Delta = 101$; a prime. Since Gauss's $p = 7$, no simple result can be expected.

$\Delta = 105 = 3 \times 5 \times 7$; (Kronecker and Weber; also *Proc. Lond. Math. Soc.*, Vol. XIX., p. 338),

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{105};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(35 \div 3)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{(21 \div 5)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6 \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(15 \div 7)}.$$

$\Delta = 109, 113, 117$, not yet solved. But 113 should have a result of same form as 41.

$\Delta = 121$; Russell puts $\lambda = \lambda' = \frac{1}{2}\sqrt{2}$,

in the modular equation of the 11th order, and obtains for

$$x = (2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}},$$

$$x^3 - 2\sqrt{2}x^2 - x - \sqrt{2} = 0.$$

$\Delta = 129$; not yet solved.

$\Delta = 133 = 7 \times 19$. According to Weber (*Math. Ann.*, Vol. XXXIII., p. 402),

$$2\kappa\kappa' = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^6 \left(\frac{5\sqrt{7}-3\sqrt{19}}{2}\right)^2, \quad \mathrm{K}'/\mathrm{K} = \sqrt{133},$$

$$2\lambda\lambda' = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^6 \left(\frac{5\sqrt{7}-3\sqrt{19}}{2}\right)^2, \quad \Lambda'/\Lambda = \sqrt{19 \div 7}.$$

$\Delta = 137, 141, 145, 149, 153, 157$, not yet solved.

$\Delta = 161 = 7 \times 23$. Putting $4\kappa\lambda\kappa'\lambda' = x^{12}$, and $x^{-1}+x = \sqrt{2}\,y$, then (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 379), the equation for y is found, by combination of the modular equations of the 7th and 23rd orders, to be

$$4y^6-32y^5+100y^4-160y^3+113y^2-24y+9 = 0,$$

or
$$(2y^2-6y+9)\left\{(25y^2-y+1)^2-23y^4\right\} = 0,$$

or
$$(2y^2-6y+9)\left\{(2y^2-5y+5)^2-23\,(y-1)^2\right\} = 0.$$

$\Delta = 165 = 3 \times 5 \times 11$; according to Weber (*Math. Ann.*, XXXIII.),

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}-\sqrt{11}}{2}\right)^2,$$

$$\mathrm{K}'/\mathrm{K} = \sqrt{(165)} = \sqrt{(3\times5\times11)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^6 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(55 \div 3)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^2,$$

$$\mathrm{K}'/\mathrm{K} = \sqrt{(33 \div 5)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^6 (3\sqrt{5}-2\sqrt{11})^2 \left(\frac{\sqrt{15}-\sqrt{11}}{2}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(15 \div 11)}.$$

$\Delta = 169 = 13^2$. According to Russell, when $\Delta = n^2$, and n is not a multiple of 3, then we obtain an equation for $z = \sqrt[12]{2\kappa\kappa'}$ by putting $\lambda = \lambda' = \frac{1}{2}\sqrt{2}$ in the modular equation of the nth degree; in this case he obtains

$$x^3-4x^2+x-12 = 0,$$

for
$$x = (2\kappa\kappa')^{-\frac{1}{12}} - (2\kappa\kappa')^{\frac{1}{12}}.$$

$\Delta = 173$; not yet solved.

$\Delta = 177 = 3 \times 59$. According to Weber (*Math. Ann.*, XXXIII.),

$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{18} \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{3}, \quad \mathrm{K}'/\mathrm{K} = \sqrt{177},$$

$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{18} \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{3}, \quad \Lambda'/\Lambda = \sqrt{(59 \div 3)}.$$

$\Delta = 193$; $(2\kappa\kappa')^{-\frac{1}{2}} + (2\kappa\kappa')^{\frac{1}{2}} = \sqrt{193} + 13$ (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 363). Obtained by approximate calculation.

$\Delta = 253 = 11 \times 23$; (Weber, *Math. Ann.*, XXXIII., p. 402),

$$2\kappa\kappa' = \left(\frac{5-\sqrt{23}}{\sqrt{2}}\right)^{6} \left(\frac{9\sqrt{23}-13\sqrt{11}}{2}\right)^{3}, \quad \mathrm{K}'/\mathrm{K} = \sqrt{253},$$

$$2\lambda\lambda' = \left(\frac{5-\sqrt{23}}{\sqrt{2}}\right)^{6} \left(\frac{9\sqrt{23}+13\sqrt{11}}{2}\right)^{3}, \quad \Lambda'/\Lambda = \sqrt{(23 \div 11)}.$$

$\Delta = 273 = 3 \times 7 \times 13$; (Weber),

$$2\kappa\kappa' = \left(\frac{\sqrt{13}-3}{2}\right)^{6} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{6} \left(\frac{15\sqrt{7}-11\sqrt{13}}{2}\right)^{2},$$

$$\mathrm{K}'/\mathrm{K} = \sqrt{(273)}.$$

$\Delta = 385 = 5 \times 7 \times 11$; (Weber),

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}-\sqrt{7}}{2}\right)^{6},$$

$$\mathrm{K}'/\mathrm{K} = \sqrt{385};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$\Lambda'/\Lambda = \sqrt{(77 \div 5)};$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}+1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$\mathrm{K}'\mathrm{K}/ = \sqrt{(55 \div 7)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^{6} \left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{6},$$

$$\Lambda'/\Lambda = \sqrt{(35 \div 11)}.$$

$\Delta = 345 = 3\times5\times23$. (Weber.)

$$2\kappa\kappa' = \left(\frac{\sqrt5-1}{2}\right)^{12}\left(\frac{\sqrt3-1}{\sqrt2}\right)^{6}\left(\frac{3\sqrt3-\sqrt{23}}{2}\right)^{6}\left(\frac{7\sqrt{23}-15\sqrt5}{\sqrt2}\right)^{2},$$

$$K'/K = \sqrt{(345)}.$$

$\Delta = 357 = 3\times7\times17$. (Weber.)

$$2\kappa\kappa' = \left(\frac{\sqrt7-\sqrt3}{2}\right)^{24}\left(\frac{3-\sqrt7}{2}\right)^{12}\left(\frac{\sqrt{21}-\sqrt{17}}{2}\right)^{6}\left(\frac{11-\sqrt{119}}{\sqrt2}\right)^{4},$$

$$K'/K = \sqrt{(357)}.$$

$\Delta = 1365 = 3\times5\times7\times13$. (Weber, *Math. Ann.*, XXXIII., p. 402.)

$$2\kappa\kappa' = \left(\frac{\sqrt5-1}{2}\right)^{6}\left(\frac{3-\sqrt7}{\sqrt2}\right)^{6}\left(\frac{\sqrt{13}-3}{2}\right)^{6}\left(\frac{\sqrt5-\sqrt3}{\sqrt2}\right)^{6}\left(\frac{\sqrt7-13}{2}\right)^{12}$$

$$\times(\sqrt{13}-2\sqrt3)^{6}\left(\frac{\sqrt{39}-\sqrt{35}}{2}\right)^{3}\left(\frac{\sqrt{65}-3\sqrt7}{\sqrt2}\right)^{2},$$

$$K'/K = \sqrt{1365};$$

the other values of $2\kappa\kappa'$ for $K'/K = \sqrt{(1365)} \div m$, where m is a factor of 1365, being obtained by an appropriate change of sign between the surds of the factors; so also for $\Delta = 273, 345, 357$, &c.

Class D; $\Delta \equiv 2$, mod. 4.

Here Hermite's invariant a is given by

$$a = -\frac{(1+\kappa^2)^4}{\kappa^2\kappa'^4};$$

and we put $\kappa'\lambda' = 2\sqrt{(\kappa\lambda)}$, equivalent to the modular equation for $n = 2$; and then introduce numbers β and γ, given by

$$\sqrt[4]{(-a)} = \beta = (\kappa\lambda)^{-\frac14} - (\kappa\lambda)^{\frac14}, \quad \gamma = \sqrt{(\beta^2+4)} = (\kappa\lambda)^{-\frac14} + (\kappa\lambda)^{\frac14}.$$

Also, if $\beta = 2\cosh 2\phi$; then

$$2\sinh 2\phi = \sqrt2\left[\left\{\tfrac12(\kappa^{-1}-\kappa)\right\}^{\frac14} + \left\{\tfrac12(\kappa^{-1}-\kappa)\right\}^{-\frac14}\right];$$

$$\kappa+\lambda = 2\sqrt{\kappa\lambda}\cosh 2\phi; \quad \kappa^{2n}+\lambda^{2n} = 2(\kappa\lambda)^n\cosh n\phi;$$

$$-\kappa+\lambda = 2\sqrt{\kappa\lambda}\sinh 2\phi; \quad -\kappa^{2n}+\lambda^{2n} = 2(\kappa\lambda)^n\sinh n\phi;$$

and then

$$e^{4\phi} = \lambda/\kappa = v^4/u^4,$$

$$\gamma = \sqrt2\left[\left\{\tfrac12(\kappa^{-1}-\kappa)\right\}^{\frac14} + \left\{\tfrac12(\kappa^{-1}-\kappa)\right\}^{-\frac14}\right],$$

in Jacobi's notation.

$\Delta = 2$: $a = -2^4$, $\beta = 2$, $\cosh\phi = 1$, $\phi = \infty$, $\kappa = \lambda = \sqrt{2}-1$.

$\Delta = 5$:

$a = -2^4/3^2$, $\beta = 2\sqrt{3}$, $\gamma = 4$, $\cosh 2\phi = \sqrt{3}$, $\sinh 2\phi = \sqrt{2}$;

$$\kappa = (\sqrt{3}-\sqrt{2})^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4, \quad \text{for } K'/K = \sqrt{5};$$

$$\lambda = (\sqrt{3}-\sqrt{2})^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4, \quad \text{for } \Lambda'/\Lambda = \sqrt{(2\div 5)}.$$

$\Delta = 10$; $\beta = 6$, $\cosh 2\phi = 3$, $\sinh\phi = 1$, $a = -2^4\times 3^4$.

$$\kappa = (\sqrt{2}-1)^2(\sqrt{10}-3), \quad \text{for } K'/K = \sqrt{10}.$$

$$\lambda = (\sqrt{2}+1)^2(\sqrt{10}-3), \quad \text{for } \Lambda'/\Lambda = \sqrt{(2\div 5)}.$$

$\Delta = 14$; then

$$a = -2^4(8\sqrt{2}+11)^2, \quad \gamma = 4\sqrt{2}+4, \quad \sqrt{(\gamma+2)} = 2+\sqrt{2};$$

$$\cosh 2\phi = \sqrt{(8\sqrt{2}+11)}, \quad \sinh 2\phi = \sqrt{(8\sqrt{2}+10)},$$

$$\kappa = \{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\}\{\sqrt{(8\sqrt{2}+11)}-\sqrt{(8\sqrt{2}+10)}\},$$

$$K'/K = \sqrt{14},$$

$$\lambda = \{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\}\{\sqrt{(8\sqrt{2}+11)}+\sqrt{(8\sqrt{2}+10)}\},$$

$$\Lambda'/\Lambda = \sqrt{(2\div 7)}.$$

$\Delta = 18$; $a = -2^4\times 7^4$,

$$\cosh 2\phi = 7, \quad \sinh 2\phi = 4\sqrt{3}, \quad \cosh\phi = 2, \quad \sinh\phi = \sqrt{3};$$

$$\kappa = (\sqrt{2}-1)^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4, \quad K'/K = \sqrt{18} = 3\sqrt{2},$$

$$\lambda = (\sqrt{2}-1)^2\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^4, \quad \Lambda'/\Lambda = \sqrt{(2\div 9)} = \tfrac{1}{3}\sqrt{2}.$$

$\Delta = 22$; $a = -\beta^4 = -2^4\times 3^4\times 11^2$, $\beta = 6\sqrt{11}$, $\gamma = 20$;

$\cosh 2\phi = 3\sqrt{11}$, $\sinh 2\phi = 7\sqrt{2}$, $\cosh 4\phi = 197$, $\sinh 4\phi = 42\sqrt{22}$;

$$\kappa = (3\sqrt{11}-7\sqrt{2})\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2, \quad K'/K = \sqrt{22};$$

$$\lambda = (3\sqrt{11}+7\sqrt{2})\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2, \quad \Lambda'/\Lambda = \sqrt{(2\div 11)}.$$

$\Delta = 26$. Here $y = \sinh\phi$ is given by the cubic equation

$$y^3+3y^2+2y+2=0 \text{ or } (y+1)^3-y+1=0.$$

$\Delta = 30$;

$$\beta = 2\sqrt{3}\,(4\sqrt{2}+5), \quad \gamma = 20+12\sqrt{2}, \quad \sqrt{(\gamma+2)} = 3\sqrt{2}+2,$$

$$\cosh 2\phi = \sqrt{3}\,(4\sqrt{2}+5), \quad \sinh 2\phi = \sqrt{10}\,(3+2\sqrt{2}),$$

$$e^{2\phi} = (\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2;$$

$$\kappa = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}-\sqrt{5})\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{30};$$

$$\lambda = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(10\div 3)};$$

$$\kappa = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}-\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{(6\div 5)};$$

$$\lambda = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2(\sqrt{3}-\sqrt{2})^2(\sqrt{6}+\sqrt{5})\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^2,$$

$$\Lambda'/\Lambda = \sqrt{(2\div 15)}.$$

$\Delta = 34$; $\quad \cosh 2\phi = 3(\sqrt{17}+4),$

$$\cosh\phi = \tfrac{1}{2}(\sqrt{17}+3) = \left(\frac{\kappa^{-1}-\kappa}{2}\right)^{\frac{1}{2}}+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{2}}.$$

$\Delta = 38$. Here β is given by the cubic equation

$$\beta^3-5\sqrt{2}\,\beta^2-2\beta-22\sqrt{2}=0.$$

$\Delta = 42$. Taking Weber's value of f_1 (*Math. Ann.*, XXXIII., p. 402), which gives

$$\tfrac{1}{2}(\kappa^{-1}-\kappa) = \tfrac{1}{8}f_1^{12} = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(2\sqrt{2}+\sqrt{7})^2,$$

we find

$$\beta = 42+16\sqrt{6}, \quad \gamma = 16\sqrt{7}+6\sqrt{42},$$

$$\cosh 2\phi = 21+8\sqrt{6}, \quad \cosh\phi = 2\sqrt{2}+\sqrt{3}, \quad \sinh\phi = \sqrt{6}+2,$$

$$\kappa = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2(\sqrt{7}-\sqrt{6})(\sqrt{2}-1)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2,$$

$$K'/K = \sqrt{42};$$

[illegible]

$$\kappa = \frac{\sqrt{2}-1}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}-\sqrt{1}}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}-\sqrt{\sqrt{2}}}}{\sqrt{2}} \cdots$$

$$k' = \sqrt{2-\sqrt{3}}$$

$$\kappa = \frac{\sqrt{2}-1}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}+\sqrt{3}}}{\sqrt{2}} \cdots$$

$$k' = \sqrt{4-\sqrt{15}}$$

$$\kappa = \frac{\sqrt{2}-\sqrt{\sqrt{3}+1}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}-\sqrt{\sqrt{3}-1}}}{\sqrt{2}} \cdot \frac{\sqrt{\sqrt{2}+\sqrt{\sqrt{3}-1}}}{\sqrt{2}} \cdots$$

$$k' = \sqrt{7-4\sqrt{3}}.$$

$\Delta = 67$, $\beta = 1.\ldots$, $\gamma = 4.\ldots - 3 = \ldots$

$$\sqrt{(\gamma-2)} = 1-\sqrt{2} \quad \left(\frac{\kappa-\kappa^{-1}}{2}\right)^{\frac{1}{2}} - \left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{2}} = 3-\ldots$$

$\Delta = 57$, [illegible] not yet solved.

$\Delta = 58$, $\beta = 166$, $\gamma = 25\sqrt{58}$ (Hermite), $\sin \frac{1}{2}\phi = 1$.

$$\kappa = (\sqrt{2}-1)^6(13\sqrt{58}-99), \quad K' : K = \sqrt{58},$$

$$\lambda = (\sqrt{2}+1)^6(13\sqrt{58}-99), \quad \Lambda' : \Lambda = \sqrt{(2+29}$$

$\Delta = 62$; $\gamma = 4\sqrt{2}+60+8\sqrt{(30\sqrt{2}+113)}$,

$$\sqrt{(\gamma+2)} = 4\sqrt{2}+4+(2+\sqrt{2})\sqrt{(4\sqrt{2}+1)}.$$

$\Delta = 66$; not yet solved.

$\Delta = 70$; $\frac{\kappa^{\frac{1}{2}}-\kappa}{2} = (\sqrt{2}+1)^6\left(\frac{\sqrt{5}+1}{2}\right)^{12}$,

$\gamma = 31878\sqrt{2}+20160\sqrt{5}$ (Weber).

$\Delta = 74$; not yet solved.

$\Delta = 78$; $\gamma = 20(13+9\sqrt{2})$, $\sqrt{(\gamma+2)} = 9\sqrt{2}+10$,

$\cosh 2\phi = \sqrt{3}(75+52\sqrt{2})$, $\sinh 2\phi = \sqrt{13}(36+25\sqrt{2}$

$$\kappa^{\frac{1}{24}} = (\sqrt{13}+2\sqrt{3})^3(3\sqrt{3}+\sqrt{26}),$$

$$\frac{\kappa^{\frac{1}{2}}-\kappa}{2} = \left(\frac{\sqrt{13}+3}{2}\right)^6(\sqrt{26}+5)^2.$$

$\Delta = 94$; $\gamma = 252+180\sqrt{2}+8\sqrt{(1422\sqrt{2}+2011)}$,

$\sqrt{(\gamma+2)} = 6\sqrt{2}+8+\sqrt{(84\sqrt{2}+118)}$.

$\Delta = 98$; not yet solved.

$\Delta = 102$; $\quad \kappa^{-1}-\kappa = 2(\sqrt{2}+1)^6(3\sqrt{2}+\sqrt{17})^4$ (Weber).

$\Delta = 106, 110, \ldots$; not yet solved.

$\Delta = 130$; $\quad \kappa^{-1}-\kappa = 2\left(\frac{\sqrt{5}+1}{2}\right)^{12}\left(\frac{\sqrt{13}+3}{2}\right)^6$ (Weber, p. 413).

$\Delta = 134, 138$; not yet solved.

$\Delta = 142$; solved by means of Russell's modular equation of the order 71 (*Proc. Lond. Math. Soc.*, Vol. XXI.). Putting

$$\kappa\lambda = x^8, \quad x^{-1}-x = v = t\sqrt{2}\,t;$$

then $\quad t^8-(4\sqrt{2}+3)t^3+(7+4\sqrt{2})t-9-6\sqrt{2} = 0,$

which has the real root $\quad t = 3\sqrt{2}+3,$

so that $\sqrt{(\gamma+2)} = \sqrt{2}\,t^3+2 = 27\sqrt{2}+38, \quad \gamma = 2900+2052\sqrt{2};$

$$\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{\frac{1}{8}}+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-\frac{1}{8}} = 9+5\sqrt{2}.$$

The following cases have also been worked out by Weber (*Math. Ann.*, Vol. XXXIII.):—

$\Delta = 82$; $\quad \left(\frac{\kappa^{-1}-\kappa}{2}\right)+\left(\frac{\kappa^{-1}-\kappa}{2}\right)^{-1} = \frac{15+\sqrt{41}}{2}.$

$\Delta = 190$; $\quad \frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^{12}(\sqrt{10}+3)^6.$

$\Delta = 210$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^6(\sqrt{3}+\sqrt{2})^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(3\sqrt{14}+5\sqrt{5}).$$

$\Delta = 330$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{5}+1}{2}\right)^{12}(\sqrt{6}+\sqrt{5})^6\left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^6(\sqrt{11}+\sqrt{10})^2.$$

$\Delta = 462$;

$$\frac{\kappa^{-1}-\kappa}{2} = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6(\sqrt{7}+\sqrt{6})^6\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^6(22\sqrt{22}+39\sqrt{7})^2.$$

[illegible]

[illegible]

[illegible]

Rotatory Polarization, illustrated by the Vibrations of a Gyrostatically loaded Chain. By J. Larmor.

[*Read June 12th*, 1890.]

The existence of rotation of the plane of polarization of light by quartz and certain organic bodies, led in Fresnel's hands to the conclusion that in these substances the waves that are propagated without change of type are circularly polarized, and that their velocity is different according as the direction in which the vibrating particles revolve round their circles is that of a right-handed or a left-handed screw.

The analytical modification of the ordinary dynamical equations of wave-propagation that would account for this peculiarity was first explicitly given by Airy.

Of strictly dynamical systems possessing the rotatory property, perhaps the only ones that have been actually realized are those founded on Sir W. Thomson's conception of *gyrostatic domination*. This idea is based on the principle that a fly-wheel, mounted without friction on an axis rigidly attached to a solid body, and initially set into rotation, modifies the dynamical properties of the solid in a peculiar and permanent manner—permanent because the fly-wheel conserves for ever the angular momentum with which it was originally endowed. A distribution of these gyrostats, scattered throughout an elastic medium, will clearly have the effect of giving rotatory properties to waves propagated in the medium.

In particular, Sir W. Thomson has considered (*Proc. Lond. Math. Soc.*, Vol. VI.) the propagation of waves of transverse displacement on a stretched chain consisting of straight links, each of which has a fly-wheel rotating on it as axis with a certain constant angular momentum.

The object here proposed is to work out in some detail the more general case in which the axes of the fly-wheels are not perpendicular to the length of the chain. We can, in this case, specify the angular momentum attached to each link by its components with reference to three axes at right angles, of which one is parallel to the chain. And it will conduce much to simplicity to proceed at once to the limit when the number of links is indefinitely great; we thus gain the idea of a uniform chain, of mass ρ per unit length, stretched with

tension r, possessed of angular momentum whose components per unit length along the chain, and in two fixed directions at right angles to it, are N, L, M.

It may be remarked that the examination of the properties of a system like this may be more than a matter of mere analytical interest. For, though the general kinematic features of light-propagation in quartz and sugar-solutions, as well as in magnetized media, are well understood, yet the nature of the dynamical mechanism by which they are produced is not known; the examination of real mechanisms possessing similar properties may therefore be useful in evolving general relations common to all such systems, and in furnishing correction to ideas that may be entertained on general grounds without a sufficient dynamical basis.

The gyrostatic systems are, of course, the analogue to the action on polarized light of magnetized media, but are not related to the rotation of a molecular character exhibited by quartz and certain organic substances.

Gyrostatic Chain.

To begin with, a change θ in the direction of the axis of a fly-wheel carrying a constant angular momentum G involves a change of the direction of this momentum, and therefore the imparting of new angular momentum $G\theta$ to the system round an axis parallel to the third side of a triangle whose other sides represent in magnitude and direction the initial and final momenta. Hence, when the change is continuous, and ω is its angular velocity, the system reacts with a couple $G\omega$, of which the axis bisects the obtuse angle between consecutive positions of the axis of the fly-wheel.

If a solid possesses gyrostatic angular momentum whose components with respect to a right-handed rectangular system of axes are L, M, N, and it is made to move with angular velocity whose components are ω_1, ω_2, ω_3, this motion is resisted by a gyrostatic couple whose components with respect to the axes are

$$M\omega_3 - N\omega_2, \quad N\omega_1 - L\omega_3, \quad L\omega_2 - M\omega_1.$$

Consider now a gyrostatically dominated stretched chain referred to an axis of z measured along its length, and axes of x and y perpendicular to its length. Let its gyrostatic angular momentum per unit length have components N, L, M with respect to these axes. Let ρ be the linear density; and let k_1, k_2, k_3 be its *effective* radii of gyration per unit length, *i.e.*, suppose each link of length δs and mass $\rho\delta s$ to have the axes of coordinates for principal axes of inertia, and the

effective principal moments of inertia with respect to the centre of mass of the link to be $\rho k_1^2 \delta s$, $\rho k_2^2 \delta s$, $\rho k_3^2 \delta s$. Let the forces at the joints between this link and the two consecutive ones have for components X, Y, Z, and $X+\delta X$, $Y+\delta Y$, $Z+\delta Z$; the only couples at the joints are those due to the torsion, which must be specified directly by means of it. The six equations of motion of the link are

$$\rho \delta s \frac{d^2x}{dt^2} = \delta X + \frac{d}{ds}\left(Z \frac{dx}{ds}\right) \delta s,$$

$$\rho \delta s \frac{d^2y}{dt^2} = \delta Y + \frac{d}{ds}\left(Z \frac{dy}{ds}\right) ds,$$

$$0 = \delta Z,$$

$$-\rho k_1^2 \delta s \frac{d^2}{dt^2}\frac{dy}{ds} = Y \delta s - \left(M \frac{d\phi}{dt} - N \frac{d^2x}{ds\,dt}\right) \delta s,$$

$$\rho k_2^2 \delta s \frac{d^2}{dt^2}\frac{dx}{ds} = -X \delta s - \left(L \frac{d\phi}{dt} + N \frac{d^2y}{ds\,dt}\right) \delta s,$$

$$\rho k_3^2 \delta s \frac{d^2\phi}{dt^2} = \frac{d}{ds}\left(\mu \frac{d\phi}{ds}\right) \delta s + \left(L \frac{d^2x}{ds\,dt} + M \frac{d^2y}{ds\,dt}\right) \delta s;$$

in which ϕ denotes the angle of twist of the chain, and μ its modulus of torsion.

When we pass to the limit, differentials are replaced by differential coefficients, and these equations become

$$\rho \frac{d^2x}{dt^2} = T \frac{d^2x}{ds^2} + \frac{dX}{ds},$$

$$\rho \frac{d^2y}{dt^2} = T \frac{d^2y}{ds^2} + \frac{dY}{ds},$$

$$Z = T,$$

$$\rho k_1^2 \frac{d^2}{dt^2}\frac{dy}{ds} = -Y + M \frac{d\phi}{dt} - N \frac{d^2x}{ds\,dt},$$

$$\rho k_2^2 \frac{d^2}{dt^2}\frac{dx}{ds} = -X + L \frac{d\phi}{dt} + N \frac{d^2y}{ds\,dt},$$

$$\rho k_3^2 \frac{d^2\phi}{dt^2} = \frac{d}{ds}\left(\mu \frac{d\phi}{ds}\right) + L \frac{d^2x}{ds\,dt} + M \frac{d^2y}{ds\,dt}.$$

The differential equations of the motion of the whole chain are now

obtained by eliminating X, Y, Z, and are

$$\rho\frac{d^2y}{dt^2}+\rho k_1^2\frac{d^2}{dt^2}\frac{d^2y}{ds^2}=T\frac{d^2y}{ds^2}+\frac{d}{ds}\left(M\frac{d\phi}{dt}-N\frac{d^2x}{ds\,dt}\right),$$

$$\rho\frac{d^2x}{dt^2}+\rho k_2^2\frac{d^2}{dt^2}\frac{d^2x}{ds^2}=T\frac{d^2x}{ds^2}+\frac{d}{ds}\left(L\frac{d\phi}{dt}+N\frac{d^2y}{ds\,dt}\right),$$

$$\rho k_3^2\frac{d^2\phi}{dt^2}=\frac{d}{d}\left(\mu\frac{d\phi}{ds}\right)+\left(M\frac{d^2y}{ds\,dt}+L\frac{d^2x}{ds\,dt}\right).$$

The simplest case will be when the breadth of the chain is very small throughout, in which case the terms involving k_1, k_2, which introduce the rotatory inertia of the chain, would be neglected, and we may write κ for k_3 in the torsional equation.*

The solution of these equations will naturally represent waves travelling along the chain. To allow them to remain of permanent type, we will take L, M, N, μ constant. To obtain the circumstances of such wave motion, we may substitute

$$(x,\ y,\ z)=(x_0,\ y_0,\ z_0)\ \text{exp.}\ \iota c\,(s-at),$$

the wave-length being $2\pi/c$, and the velocity of propagation a. Thus

$$\left.\begin{aligned}-\iota\frac{c}{a}Nx_0+\left(\frac{T}{a^2}+\rho k_1^2c^2-\rho\right)y_0+\frac{1}{a}M\phi_0=0,\\ \left(\frac{T}{a^2}+\rho k_2^2c^2-\rho\right)x_0+\iota\frac{c}{a}Ny_0+\frac{1}{a}L\phi_0=0,\\ Lx_0+My_0+\left(\frac{\mu}{a}-\rho k_3^2a\right)\phi_0=0.\end{aligned}\right\}$$

Hence we have for a the equation

$$\begin{vmatrix}-\iota\frac{c}{a}N, & \frac{T}{a^2}+\rho k_1^2c^2-\rho, & M\\ \frac{T}{a^2}+\rho k_2^2c^2-\rho, & \iota\frac{c}{a}N, & L\\ L, & M, & \mu-\rho k_3^2a^2\end{vmatrix}=0;$$

or $$\left(\frac{\mu}{a^2}-\rho k_3^2\right)\left\{\left(\frac{T}{a^2}+\rho k_1^2c^2-\rho\right)\left(\frac{T}{a^2}+\rho k_2^2c^2-\rho\right)-\frac{c^2}{a^2}N^2\right\}$$
$$-\frac{L^2}{a^2}\left(\frac{T}{a^2}+\rho k_1^2c^2-\rho\right)-\frac{M^2}{a^2}\left(\frac{T}{a^2}+\rho k_2^2c^2-\rho\right)=0.$$

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* *Cf.* Mathematical Tripos, Part II., 1890 (*Camb. Exam. Papers*, 1890, p. 585).

This is an equation of the third degree in $1/a^2$, so that there are three types of motion.

When there is no gyrostatic element, there are two transverse waves, and one torsional wave. If the reciprocals of their velocities are u_1, u_2, u_3, we have

$$\frac{Tu_1^2}{\rho} = 1-\left(\frac{2\pi k_1}{\lambda}\right)^2, \quad \frac{Tu_2^2}{\rho} = 1-\left(\frac{2\pi k_2}{\lambda}\right)^2, \quad \frac{\mu u_3^2}{\rho} = k_3^2.$$

If u denote $1/a$, the general velocity equation is therefore

$$\mu T^2(u^2-u_1^2)(u^2-u_2^2)(u^2-a_3^2) - L^2Tu^2(u^2-u_1^2) - M^2Tu^2(u^2-u_2^2) - c^2N^2\mu u^2(u^2-u_3^2) = 0.$$

This equation determines the character of the motions. We notice by the signs of the left-hand side that its three roots are real and positive, one lower than the least of u_1, u_2, u_3, one higher than the greatest, and one intermediate between them. Thus, T being positive, a disturbance usually starts three waves, which are propagated along the chain both ways.

It may even happen that when T is negative the roots are all positive, provided the gyrostatic terms are great enough. A finite piece of the perfectly flexible chain may then be subjected to end-thrust, and yet will not collapse unless the thrust exceeds a certain limit—a noteworthy example of gyrostatically conferred rigidity.

In the general case, the ratios $x_0 : y_0 : \phi_0$ will usually be complex quantities; hence, in any of the three types of undulation, the links of the chain will describe similar ellipses transverse to its length, and the wave will be elliptically polarized.

In the case which corresponds to the rotation of polarized light, the effect of the gyrostatic influence will be to modify only slightly the circumstances of the waves produced by tension and elasticity. The nature of the change can, if necessary, be examined by an obvious method of approximation.

We proceed to review some simple cases of the general result.

(i.) $L = 0$, $M = 0$. Here the torsional wave is unaffected; and the velocities of the other two are given by

$$\mu T^2(u^2-u_1^2)(u^2-u_2^2) - c^2N^2\mu u^2 = 0,$$

or

$$u^4 - (u_1^2+u_2^2+c^2N^2/T^2)u^2 + u_1^2u_2^2 = 0.$$

The roots are both real and positive, even when T is negative. The

chain will therefore remain straight under any end-thrust, however great, provided it is applied exactly along the chain; a result we are prepared for by the behaviour of an ordinary top.

The value of $x_0 : y_0$ shows that the transverse vibrations are elliptically polarized. The polarization will, of course, become circular when $k_1 = k_2$; but the velocity of propagation will be different according as the rotation is right-handed or left-handed.

(ii.) $N = 0$, $L = M$, $k_1 = k_3$, so that $u_1 = u_3$. Here

$$u = u_1,$$

or
$$\mu T (u^2 - u_1^2)(u^2 - u_2^2) - L^2 u^2 = 0,$$

that is,
$$u^4 - (u_1^2 + u_3^2 + L^2/\mu T)\, u^2 + u_1^2 u_3^2 = 0.$$

The roots of this latter are real and positive, even when T is negative, provided $(u_1 - u_3)^2$ numerically exceed $L^2/\mu T$, that is, if T exceeds $L^2/\mu\,(u_1 - u_3)^2$. So far as these vibrations are concerned, the chain could resist thrusts beyond a certain value, but would collapse under smaller ones; this anomaly will not however occur, for the vibrations of the type corresponding to $u = u_1$ give instability for all thrusts.

The root $u = u_1$ corresponds to a plane-polarized flexural vibration unaccompanied by torsion. The other roots correspond each to a plane-polarized flexural vibration accompanied by a torsional one of the same phase.

(iii.) $M = 0$, $N = 0$. Here

$$u = u_1,$$

or
$$\mu T (u^2 - u_2^2)(u^2 - u_3^2) - L^2 u^2 = 0,$$

which may be discussed as in case (ii.).

The root $u = u_1$ corresponds to a plane-polarized vibration in the plane containing the gyrostats. The other roots correspond each to a plane-polarized vibration perpendicular to the plane of the gyrostats, accompanied by a torsional vibration of the same phase.

Reflexion of Circularly Polarized Waves.

The circumstances of the passage of light through a piece of magnetized heavy-glass may be illustrated by the passage of a circularly polarized wave through a piece of gyrostatic chain of length l inserted in a chain free from gyrostatic influence. The equations of vibration

for this part are those of case (i.) above, viz.,

$$\frac{d^2x}{dt^2} = a^2\frac{d^2x}{ds^2} + c\frac{d^2y}{ds^2dt},$$

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{ds^2} - c\frac{d^2x}{ds^2dt},$$

If we introduce the complex variable

$$w = x + \iota y,$$

these equations are both included in the form

$$\frac{d^2w}{dt^2} = a^2\frac{d^2w}{ds^2} - \iota c\frac{d^2w}{ds^2dt}.$$

An incident right-handed circularly polarized series of waves is represented by

$$w = A_1 \exp \iota (nt - ex).$$

Let us trace its progress: we have in the successively traversed parts of the chain

$$w_1 = A_1 \exp \iota (nt - ex) + B_1 \exp \iota (nt + ex),$$

$$w_2 = A' \exp \iota (nt - e'x) + B' \exp \iota (nt + e'x),$$

$$w_3 = A_3 \exp \iota (nt - ex);$$

for in the parts (1) and (2) there is a reflected wave travelling backward, as well as the direct one.

Substitution in the equation of motion gives

$$e = n/a$$

(or say n/a, if the a were not the same for both parts)

$$e' = n/(a^2 + cn)^{\frac{1}{2}}.$$

The conditions necessary for continuity are—

at $x = 0$, $\quad w_1 = w_2, \quad \frac{dw_1}{dx} = \frac{dw_2}{dx}$;

at $x = l$, $\quad w_2 = w_3, \quad \frac{dw_2}{dx} = \frac{dw_3}{dx}$;

giving

$$\left.\begin{aligned} A_1 + B_1 &= A' + B', \\ (A_1 - B_1)\, e &= (A' - B')\, e', \end{aligned}\right\}$$

and
$$\left.\begin{aligned} A_3 \exp(-\iota el) &= A' \exp(-\iota e'l) + B' \exp(\iota e'l), \\ eA_3 \exp(-\iota el) &= e' \{A' \exp(-\iota e'l) - B' \exp(\iota e'l)\} \end{aligned}\right\}.$$

Thus
$$2A_1 = A'\left(1 + \frac{e'}{e}\right) + B'\left(1 - \frac{e'}{e}\right),$$

so that
$$2\frac{A_1}{A_3} = \tfrac{1}{2}\exp(-\iota el)\left\{\left(\frac{e}{e'} + \frac{e'}{e} + 2\right)\exp(\iota e'l) - \left(\frac{e}{e'} + \frac{e'}{e} - 2\right)\exp(-\iota e'l)\right\}$$
$$= C\exp(-\iota el + \iota\kappa);$$

where
$$\left.\begin{aligned} C\sin\kappa &= \left(\frac{e}{e'} + \frac{e'}{e}\right)\sin e'l \\ C\cos\kappa &= 2\cos e'l \end{aligned}\right\}.$$

Hence
$$\frac{A_3}{A_1} = \frac{2}{C}\exp(\iota el - \iota\kappa).$$

Therefore w_3 represents a right-handed circularly polarized system of waves, with amplitude $2A_1/C$, and acceleration of phase $el - \kappa$, where

$$\tan\kappa = \tfrac{1}{2}\left(\frac{e}{e'} + \frac{e'}{e}\right)\tan e'l$$
$$= \frac{a^2 + \frac{1}{2}cn}{a(a^2+cn)^{\frac{1}{2}}}\tan\frac{nl}{(a^2+cn)^{\frac{1}{2}}}.$$

Consider now an incident left-handed system of waves. Its equation is
$$x - \iota y = A_1 \exp\iota(nt - ex),$$
so that we must employ the independent variable $u = x - \iota y$, and the equation of motion is $\dfrac{d^2u}{dt^2} = a^2\dfrac{d^2u}{ds^2} + \iota c\dfrac{d^3u}{ds^2\,dt}$.

The results may therefore be derived from the case already worked out by changing the sign of c.

Finally, an incident system of plane-polarized waves may be split up into two equal systems of right-handed and left-handed waves, each of which, as has been seen, will remain of the same type during transmission.

The emergent waves w_3 and u_3 will be of different amplitudes, and their accelerations of phase will be different. If they had the same amplitude, they would combine into a plane-polarized wave rotated through an angle equal to half the difference of their accelerations of phase. But, the amplitudes being different, they actually combine

into an elliptically polarized wave with the major axes of its vibrations rotated through this angle.

In this general case the emergent vibration is not plane-polarized, nor is the rotation proportional to the thickness of the gyrostatic medium. In the optical problem, however, the rotatory terms are very small compared with the others. And it will be found, on making c small, that to the *second* order of small quantities included, the results agree with the ordinary optical rule, which makes the emergent light plane-polarized and the rotation proportional to the thickness of the rotating medium and inversely as the square of the wave-length.

The reason of this agreement is that the values of a have been taken the same in both media, so that the only difference between them is the slight one denoted by the rotational coefficient c. The intensity of the reflexion is therefore very small, and the circular waves may be treated as if they passed through in unchanged type without any reflexion, and were finally compounded by the kinematic rule. This is the mode of explanation usually employed to illustrate or account for the optical phenomena. But strictly the velocities of propagation are very different—for example, in glass and air; and if we write $e = n/a$ instead of n/a in the above analysis, we are led to a different result. In fact, the successive reflexions backward and forward at the ends of the optical plate have now a material influence, and in an exact investigation must be taken into account, either by a method analogous to the above, or by the mode of calculation usually adopted in the exact theory of interferences by reflexions at the surfaces of a thin plate.

In experimental investigations with thick plates, the influence of the reflected light on the result is usually avoided by a slight inclination of the rays, which separates the reflected from the direct part.*

The influence of the reflected light, when it is not thus eliminated, is, in fact, that we would have approximately the series

$$\exp \iota \left(\frac{2\pi t}{\tau} - \frac{2\pi l}{\lambda} + \gamma l\right) + \epsilon^2 \exp \iota \left(\frac{2\pi t}{\tau} - \frac{6\pi l}{\lambda} + 3\gamma l\right)$$
$$+ \epsilon^4 \exp \iota \left(\frac{2\pi t}{\tau} - \frac{8\pi l}{\lambda} + 5\gamma l\right) + \ldots$$

to represent the emergent undulation, where ϵ^2 is the coefficient of intensity of a single reflexion.

This sums to
$$\frac{\exp \iota \left(\frac{2\pi t}{\tau} - \frac{2\pi l}{\lambda} + \gamma l\right)}{1 - \epsilon^2 \exp \iota \left(-\frac{4\pi l}{\lambda} + 2\gamma l\right)},$$

* *Cf.* Lord Rayleigh, *Phil. Trans.*, II., 1885, p. 345.

or, when ϵ^4 is neglected,

$$\exp \iota \left\{ \frac{2\pi t}{\tau} - \frac{2\pi l}{\lambda}(1+2\epsilon^2) + \gamma l (1+2\epsilon^2) \right\}.$$

Thus the coefficient of rotation would be increased in the ratio $1+2\epsilon^2$ to unity.

On certain Concomitants of Systems of Conics and Quadrics, and on the Calculation of the Covariant S of the Ternary Quartic.

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[*Read June 12th*, 1890.]

The following paper is divided into three parts. In Part I., several concomitants of a system of line-pairs, conics, or quadric surfaces are calculated by appropriate methods. The expressions for several of these are believed to be given now for the first time. The result of § 6 was given, however, in *Wiskundige Opgaven*, 1888, *Voorstel* 143, but was arrived at by a different process. Part II. consists of two involution problems, and Part III. is concerned with the calculation of the covariant S of the ternary quartic.

Part I.

On the Calculation of certain Concomitants of a System of Three Quadrics in terms of the Fundamental Concomitants of the System.

1. When the fundamental invariants of a system of quantics of the same degree have been found, the combinant invariants are easily found in some cases, where the combinant contains the different sets of coefficients symmetrically. For we have only to write down the symmetrical groups of invariants containing the coefficients in the required degree, to add these affected by unknown multipliers, then to differentiate the sum with respect to the members of one set of coefficients, multiplying at the same time by the corresponding members of another set, and lastly to add these results and to equate the coefficients of the different terms of the sum to zero. We shall have then, in general, more than enough equations to determine the multipliers.

2. Let three m-ary quadrics be denoted by (1), (2), and (3); let their discriminants be denoted by (1^m), (2^m), (3^m), and let ${}_2\partial_1$ denote the result of adding the differential coefficients with respect to the coefficients of (1) multiplied by the corresponding coefficients of (2).

The fundamental invariants of the system are

$$(1^m),\ (2^m),\ (3^m);$$

$3m$ invariants of the types

$$(2^r3^s) \equiv \frac{1}{\lfloor r} ({}_2\partial_3)^r (3^m) \equiv \frac{1}{\lfloor s} ({}_2\partial_3)^s (2^m),$$

$$(3^r1^s) \equiv \frac{1}{\lfloor r} ({}_3\partial_1)^r (1^m) \equiv \frac{1}{\lfloor s} ({}_1\partial_3)^s (3^m),$$

$$(1^r2^s) \equiv \frac{1}{\lfloor r} ({}_1\partial_2)^r (2^m) \equiv \frac{1}{\lfloor s} ({}_2\partial_1)^s (1^m),$$

where r and s are to take all positive integral values subject to the relation $r+s=m$; and

$$\frac{m\,(m-1)(m-2)}{3!}$$

invariants of the type

$$(1^r2^s3^t) \equiv \frac{1}{\lfloor s \lfloor t} ({}_2\partial_1)^s ({}_3\partial_1)^t (1^m),$$

where r, s, and t are to take all positive integral values consistent with the relation $r+s+t=m$. It is seen that

$${}_1\partial_3 (1^r2^s3^t) = (r+1)(1^{r+1}2^s3^{t-1}),$$

except when $t=0$, and is then zero.

3. *To express in terms of the fundamental invariants of a system of three line-pairs, the invariant that must vanish if the three are in involution.*

We start with the knowledge that the invariant contains the coefficients in the first degree, and is a combinant symmetrical in the different sets of coefficients, and that it is impossible to express any symmetrical combinant, of the first degree in the coefficients, in terms of the fundamental invariants. Let us then find the symmetrical combinant involving the coefficients in the second degree. This must be of the form

$$A\,(1^2)\,(2^2)\,(3^2) + B\{(1^2)\,(23)^2 + (2^2)\,(31)^2 + (3^2)\,(12)^2\} + C\,(23)\,(31)\,(12) \quad \ldots\ldots\ldots\ (1).$$

Operating ${}_1\partial_3$ on this, we have

$$A\,(1^2)\,(2^2)\,(13) + B\{2\,(1^2)\,(12)\,(23) + 4\,(2^2)\,(1^2)\,(13) + (13)\,(12)^2\} + C\{(12)^2\,(31) + 2\,(1^2)\,(23)\,(12)\}.$$

Equating the coefficients of the different terms in this result to zero, we find

$$A = 4, \quad B = -1, \quad C = 1.$$

The result of substituting these values in (1) is the only symmetrical combinant containing the coefficients in the second degree. It is, therefore, the square of the combinant we are seeking. Let the combinant be denoted by C; then

$$C^2 = 4(1^2)(2^2)(3^2) - \{(1^2)(23)^2 + (2^2)(31)^2 + (3^2)(12)^2\} + (23)(31)(12).$$

The full working has been given in this instance, in order to illustrate the method. In arriving at the following results, the same method has been employed, but the working is omitted.

4. *Expression, in terms of the fundamental invariants of a system of three solid quadrics, of the invariant that must vanish if the system of quadrics have a common harmonic pentahedron.*

Let C denote the symmetrical combinant in question. It is found that

$$\begin{aligned} C^2 = {} & 144(1^4)(2^4)(3^4) - 36\{(1^4)(2^23)(23^2)\} \\ & + 12\{(1^4)(2^23^2)^2\} + 9\{(1^22)(12^23)(23^2)\} \\ & - 6\{1^22)(123^2)(2^23^2)\} + 9\{(1^22)(13^2)(2^23)\} \\ & + 4\{(1^22^2)(1^23^2)(2^23^2)\} + 2\{(1^223)^2(2^23^2)\} \\ & - (1^223)(12^23)(123); \end{aligned}$$

where each term within parentheses stands for the sum of all the *different* terms obtained from it by interchanging the symbols 1, 2, and 3, in all possible ways.

5. *Expression, in terms of the fundamental invariants of a system of three conics, of the condition that they should meet in a point.*

Let R be the resultant of the system. Then

$$\begin{aligned} R = {} & (123)^4 - 8(123)^2\{(12^2)(13^2)\} - 72(123)(1^3)(2^3)(3^3) \\ & + 24(123)\{(1^3)(23^2)(32^2)\} + 16(123)\{(1^22)(2^23)(3^21\} \\ & + 48\{(2^3)(3^3)(1^22)(1^23)\} - 48\{(1^3)(12^2)(23^2)^2\} \\ & + 16\{(12^2)^2(13^2)^2\} - 16\{(1^22)(1^23)(2^23)(23^2)\}; \end{aligned}$$

where each term within parentheses stands for the sum of all terms obtained from it by interchanging the symbols 1, 2, and 3 in all possible ways.

6. *To express, in terms of the fundamental covariants and invariants of a system of three conics, the covariant which equated to zero gives the locus of points, the tangents from which to the three conics form a system in involution.*

Let the point equations to the conics, as formed in the usual way from their line equations, be

$$s_{11} = 0, \quad s_{22} = 0, \quad s_{33} = 0;$$

and let

$$s_{23} = 0, \quad s_{31} = 0, \quad s_{12} = 0,$$

be the equations to the director conics, as found in an analogous manner. Then, with regard to the points in which the tangent pairs to the conics from any point cut any line $l = 0$, we have, in terms of the coordinates of the point,

$$(1^2) = l^2 s_{11}, \quad (2^2) = l^2 s_{22}, \quad (3^2) = l^2 s_{33},$$

$$(23) = l^2 s_{23}, \quad (31) = l^2 s_{31}, \quad (12) = l^2 s_{12}.$$

Substituting in the result of § 3, and dividing by l^6, we have

$$I^2 = 4 s_{11} s_{22} s_{33} - \left\{ s_{11} s_{23}^2 + s_{22} s_{31}^2 + s_{33} s_{12}^2 \right\} + s_{23} s_{31} s_{12};$$

$I = 0$ being the equation to the locus of points, the tangents from which to the three conics form a system in involution.

7. *To find the equation to the scroll generated by the lines touching each of three quadrics.*

If a point be on the scroll whose equation we are seeking, the three tangent-cones from the point to the three quadrics have a common generator, and the section of the cones by any plane have, therefore, a common point.

Let the equations to the quadrics be

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0,$$

and let ${}_1S_2$ denote

$$(b_1 c_1 d_1 + 2 f_1 m_1 n_1 - \&c.)(a_2 x + h_2 y + g_2 z + l_2 w)^2 + \&c.,$$

employing the usual notation, so that ${}_1S_2 = 0$ is the equation of the polar reciprocal of (1) with regard to (2).

Now, with regard to the sections of the tangent-cones by any plane $p = 0$, we have

$$(1^3) = p^3 (1^4)\, u_1^2,$$

$$(1^2 2) = p^2 \{(1^2 2)\, u_1 - {}_1S_2\}\, u_1,$$

$$\begin{aligned}(123) = p^2 \big[&\tfrac{1}{1\cdot 2} \{(2^2 3^2)\, u_1^2 + (3^2 1^2)\, u_2^2 + (1^2 2^2)\, u_3^2\} \\ &- \tfrac{1}{3}\{(1^2 23)\, u_2 u_3 + (12^2 3)\, u_3 u_1 + (123^2)\, u_1 u_2\} \\ &+ \tfrac{1}{4}\{u_1 ({}_1\partial_{3\,3} S_2 + {}_1\partial_{2\,2} S_3) + u_2 ({}_2\partial_{3\,3} S_1 + {}_2\partial_{1\,1} S_3) \\ &+ u_3 ({}_3\partial_{2\,2} S_1 + {}_3\partial_{1\,1} S_2)\} \big].\end{aligned}$$

The equation to the scroll will therefore be found by substituting in R these expressions, after suppressing the irrelevant factor p^2, and equating the result to zero.

Part II.

Two Involution Problems.

1. *To find the equation to the envelope of lines cut in involution by three conics.*

The condition that three pencils, whose equations written symbolically are

$$(a_1 x_1 + a_2 x_2)^2 = 0, \quad (a_1' x_1 + a_2' x_2)^2 = 0, \quad (a_1'' x_1 + a_2'' x_2)^2 = 0,$$

should be in involution is

$$(a_1' a_2'' - a_2' a_1'')(a_1'' a_2 - a_2'' a_1)(a_1 a_2' - a_2 a_1') = 0 \ldots\ldots\ldots\ldots (1).$$

Let the three conics have for their equations

$$(a_1 x_1 + a_2 x_2 + a_3 x_3)^2 = 0,$$

$$(a_1' x_1 + a_2' x_2 + a_3' x_3)^2 = 0,$$

$$(a_1'' x_1 + a_2'' x_2 + a_3'' x_3)^2 = 0,$$

and let $y_1 y_2 y_3$, $z_1 z_2 z_3$, be two points on the line whose coordinates are $\xi_1 \xi_2 \xi_3$. Let the line joining the two points be cut by a conic in the ratio $\kappa_2 : \kappa_1$. Then the ratios for the three conics are determined from the equations

$$\{\kappa_1 (a_1 y_1 + a_2 y_2 + a_3 y_3) + \kappa_2 (a_1 z_1 + a_2 z_2 + a_3 z_3)\}^2 = 0,$$

$$\{\kappa_1 (a_1' y_1 + a_2' y_2 + a_3' y_3) + \kappa_2 (a_1' z_1 + a_2' z_2 + a_3' z_3)\}^2 = 0,$$

$$\{\kappa_1 (a_1'' y_1 + a_2'' y_2 + a_3'' y_3) + \kappa_2 (a_1'' z_1 + a_2'' z_2 + a_3'' z_3)\}^2 = 0.$$

Consequently, to find the condition sought, we must in (1) substitute

$$a_1y_1+a_2y_2+a_3y_3 \text{ for } a_1, \quad a_1z_1+a_2z_2+a_3z_3 \text{ for } a_2,$$

and so on.

On substituting, $a_1'a_2''-a_2'a_1''$ becomes

$$\begin{aligned}&(a_1'y_1+a_2'y_2+a_3'y_3)(a_1''z_1+a_2''z_2+a_3''z_3)\\ &\qquad -(a_1''y_1+a_2''y_2+a_3''y_3)(a_1'z_1+a_2'z_2+a_3'z_3)\\ &= (y_2z_3-y_3z_2)(a_2'a_3''-a_3'a_2'')+(y_3z_1-y_1z_3)(a_3'a_1''-a_1'a_3'')\\ &\qquad +(y_1z_2-y_2z_1)(a_1'a_2''-a_2'a_1'')\\ &= \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ a_1', & a_2', & a_3' \\ a_1'', & a_2'', & a_3'' \end{vmatrix}.\end{aligned}$$

The other factors of (1) may be treated similarly. Thus, the symbolical equation to the envelope of lines cut in involution by the three conics is

$$\begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ a_1', & a_2', & a_3' \\ a_1'', & a_2'', & a_3'' \end{vmatrix} \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ a_1'', & a_2'', & a_3'' \\ a_1, & a_2, & a_3 \end{vmatrix} \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ a_1, & a_2, & a_3 \\ a_1', & a_2', & a_3' \end{vmatrix} = 0.$$

If the equations to the conics be written in the forms

$$(abcfgh\,\mathfrak{J}\,xyz)^3=0, \quad (a'b'c'f'g'h'\,\mathfrak{J}\,xyz)^3=0, \quad (a''b''c''f''g''h''\,\mathfrak{J}\,xyz)^3=0,$$

and the coordinates of any line be represented by l, m, n, the equation to the envelope sought is

$$\begin{aligned}&(bcf)\,l^3+(cag)\,m^3+(abh)\,n^3+\{-(bcg)+2\,(hfc)\}\,l^2m\\ &\quad+\{-(bch)+2\,(fgb)\}\,l^2n+\{-(cah)+2\,(fga)\}\,m^2n\\ &\quad+\{-(caf)+2\,(ghc)\}\,m^2l+\{-(abf)+2\,(ghb)\}\,n^2l\\ &\quad+\{-(abg)+2\,(hfa)\}\,n^2m+\{(abc+4\,(fgh)\}\,lmn=0;\end{aligned}$$

where any expression such as (pqr) denotes the determinant

$$\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}.$$

2. *To find the complex of lines cut in involution by three quadrics.*

Let $y_1y_2y_3y_4$, $z_1z_2z_3z_4$ be two points on the line whose coordinates

are $\xi\eta\zeta\xi'\eta'\zeta'$. Then, if the line joining the two points is cut by one of the quadrics $a_x^2 = 0$, $a_x'^2 = 0$, $a_x''^2 = 0$, in the ratio $\kappa_2 : \kappa_1$, we have, for the determination of the three pairs of ratios, the three equations

$$(\kappa_1 a_y + \kappa_2 a_z)^2 = 0,$$
$$(\kappa_1 a_y' + \kappa_2 a_z')^2 = 0,$$
$$(\kappa_1 a_y'' + \kappa_2 a_z'')^2 = 0.$$

To get the condition that the line should be cut in involution by the three quadrics, we must therefore substitute, in (1), a_y for a_1, a_z for a_2, and so on. The factor $a_1'a_2'' - a_2'a_1''$ then becomes

$$\begin{aligned}&(y_3z_3 - y_3z_1)(a_2'a_3'' - a_3'a_2'') + (y_3z_1 - y_1z_3)(a_3'a_1'' - a_1'a_3'')\\ &\quad + (y_1z_2 - y_2z_1)(a_1'a_2'' - a_2'a_1'') + (y_1z_4 - y_4z_1)(a_1'a_4'' - a_4'a_1'')\\ &\quad + (y_2z_4 - y_4z_2)(a_2'a_4'' - a_4'a_2'') + (y_3z_4 - y_4z_3)(a_3'a_4'' - a_4'a_3'')\\ &= \xi(a_2'a_3'') + \eta(a_3'a_1'') + \zeta(a_1'a_2'') + \xi'(a_1'a_4'') + \eta'(a_2'a_4'') + \zeta'(a_3'a_4'').\end{aligned}$$

The symbolical equation to the complex sought is therefore

$$\begin{aligned}&\{\xi(a_2'a_3'') + \eta(a_3'a_1'') + \zeta(a_1'a_2'') + \xi'(a_1'a_4'') + \eta'(a_2'a_4'') + \zeta'(a_3'a_4'')\}\\ &\times\{\xi(a_2''a_3) + \eta(a_3''a_1) + \zeta(a_1''a_2) + \xi'(a_1''a_4) + \eta'(a_2''a_4) + \zeta'(a_3''a_4)\}\\ &\times\{\xi(a_2a_3') + \eta(a_3a_1') + \zeta(a_1a_2') + \xi'(a_1a_4') + \eta'(a_3a_4') + \zeta'(a_2a_4')\} = 0.\end{aligned}$$

Part III.

Calculation of the Covariant S of the Ternary Quartic.

The calculation in full of the covariant S of the ternary quartic given below, was effected in a manner that seems very appropriate for the calculation of covariants which are obtained by forming invariants of the polars of a point. The covariant might, of course, be derived by simply substituting, in the expression for the invariant, the values of the coefficients in terms of the coordinates of the point, and grouping together like terms; but this method seems less convenient than the one here employed.

The quartic is written in the form

$$\begin{aligned}&ax^4 + by^4 + cz^4 + 4x^3(a_2y + a_3z) + 4y^3(b_1x + b_3z)\\ &\qquad + 4z^3(c_1x + c_2y) + 6(fy^2z^2 + gz^2x^2 + hx^2y^2)\\ &\qquad + 12xyz(lx + my + nz).\end{aligned}$$

To find the coefficient of any term—x^2yz say—of the covariant, transfer to each of the terms of the invariant S of the cubic (Salmon, *H. P. C.*, 3rd ed., p. 191), the suffixes 1, 1, 2, 3 in all possible ways,

and interpret the different terms so obtained by the aid of the ccompanying table.

	a	a_2	a_3	b_1	b	b_3	c_1	c_2	c	m
1	a	a_2	a_3	h	b_1	m	g	n	c_1	b
2	a_2	h	l	b_1	b	b_3	n	f	c_2	m
3	a_3	l	g	m	b_3	f	c_1	c_2	c	a

The top row of the table contains the coefficients of the cubic, and the extreme left-hand column the suffixes.

Thus, we get from the term $abcm$, when working out the coefficient of x^2yz, 12 terms by attaching the suffixes in the orders 2113, 2131, 2311, 3112, 3121, 3211, 1123, 1132, 1213, 1312, 1231, 1321; and on interpreting, we get

$$a_3b_1c_1n+a_3b_1cl+a_2b_3c_1l+a_3b_1c_1m+a_3b_1c_2l+a_2bc_1l+ab_1c_3n$$

$$+ab_1cm+abc_1n+ab_3c_1m+abc_2l+ab_3c_2l.$$

From the term $-mt^4$, we get $-(mlln+mlnl+\&c.)$; *i.e.*, $-12l^2mn$.

In like manner, when calculating the value of the coefficient of x^4, we attach the suffixes 1, 1, 1, 1 to the letters in each of the terms of the invariant S. Thus, we get from the term $+ab_1c_2^2$ the term $+ahn^2$.

The proof of this principle is almost obvious. For, let the quartic in symbolical notation be written in any of the forms a^4, b^4, c^4, d^4. Then, if A denote any term of the invariant of the cubic written symbolically, the corresponding term of the invariant of the quartic will be $a_xb_xc_xd_xA$. To get the coefficient of x_1^4 from this, we have to change A into $a_1b_1c_1d_1A$. Now A is of the form

$$a_1^ra_2^sa_3^t\,.\,b_1^{r'}b_2^{s'}b_3^{t'}\,.\,c_1^{r''}c_2^{s''}c_3^{t''}\,.\,d_1^{r'''}d_2^{s'''}d_3^{t'''},$$

where $r+s+t=r'+s'+t'=r''+s''+t''=r'''+s'''+t'''=3.$

The corresponding coefficient of x_1^4, derived from the term A, will therefore be

$$a_1^{r+1}a_2^sa_3^t\,.\,b_1^{r'+1}b_2^{s'}b_3^{t'}\,.\,c_1^{r''+1}c_2^{s''}c_3^{t''}\,.\,d_1^{r'''+1}d_2^{s'''}d_3^{t'''},$$

and this can be expressed at once in terms of the coefficients of the quartic. But, if

$$a_1^r a_2^s a_3^t = l, \quad b_1^{r'} b_2^{s'} b_3^{t'} = m, \quad c_1^{r''} c_2^{s''} c_3^{t''} = n, \quad d_1^{r'''} d_2^{s'''} d_3^{t'''} = p,$$

we may denote $a_1 l$ by l_1, $b_1 m$ by m_1, $c_1 n$ by n_1, $d_1 p$ by p_1; and in the same way $a_2 l$ by l_2, $a_3 l$ by l_3, &c.; and then, having tabulated l_1, m_1, n_1, l_2, l_3, &c., we can at once interpret terms such as $l_1 m_1 n_1 p_1$. So, too, the term in $x_1^2 x_2 x_3$, derived from A, will be $A \times$ sum of all possible combinations, from the groups $a_1 b_1 c_1 d_1$, $a_2 b_2 c_2 d_2$, $a_3 b_3 c_3 d_3$, that contain two letters from the first group, and one from each of the other two, and we shall get for the coefficient of $x_1^2 x_2 x_3$ the result of attaching the suffixes 1, 1, 2, 3 to the terms of the product $lmnp$ in all possible ways, and then adding the results.

With a machine derived from an obvious modification of a type-writer, the coefficients of any of this class of covariant could be rapidly calculated, and the machine could be made effective for other than ternary quantics.

The following is the expression in full of typical terms of the covariant S, calculated in the manner indicated above:—

$$\begin{aligned}
&x^4 \big[-a_3^2 a_2 b_1 c_1 + a_3 b_1 g^3 + a_3 c_1 h^3 - g^3 h^3 \\
&+ \{ a b_1 c_1 - a_3 b_1 g - a_2 c_1 h \} l + \{ a_3^2 c_1 - a c_1 h + a_3 g h \} m \\
&\qquad + \{ a_3^2 b_1 - a b_1 g + a_2 g h \} n \\
&+ 2 g h l^3 - \{ a_3^2 - a g \} m^3 - \{ a_2^2 - a h \} n^3 + a_2 a_3 m n - 3 a_3 h n l - 3 a_2 g l m \\
&+ 2 a_3 l^3 m + 2 a_2 l^3 n - a l m n \\
&- l^4 \big], \\
&+ y^4 \ldots\ldots \qquad + z^4 \ldots\ldots . \\
&+ x^3 y \\
&\quad \big[a_3^2 b_2 c_1 - a_2 a_3 (b c_1 + b_1 c_2) + a_3^2 b_1 f - (a b_3 - a_3 b_1)\, c_1 h + a_2 b g^2 + a_3 c_2 h^2 \\
&\qquad + a_3 b_3 g h - a b_1 f g - b_1 g^2 h + a_2 f g h \\
&\quad - \{ -a b c_1 - a b_1 c_2 + a_2 b_1 c_1 + (a_3 b + 3 a_2 b_3)\, g + a_2 c_1 h + 3 a_3 f h \}\, l \\
&\quad + \{ a_3^2 c_2 - 2 a_3^2 b_3 + a_2 a_3 f + 2 a b_3 g - a c_2 h \}\, m + \{ a_2 a_3 b_3 + a_3^2 b - 2 a_3^2 f \\
&\qquad - (a b - 2 a_2 b_1)\, g + 2 a h f - g h^2 \}\, n \\
&\quad + \{ 2 a_3 b_3 + 2 a_2 f + b_1 g \}\, l^2 - 2 a_2 g m^2 - a_3 h m n - (a b_3 + 2 a_3 b_1)\, n l \\
&\qquad - (a f - 2 g h)\, l m \\
&\quad + h l^2 n + 2 a_3 m^2 l + a_2 l m n \\
&\quad - 2 l^3 m \big], \\
&+ x^3 z \ldots \quad + y^3 x \ldots \quad + y^3 z \ldots \quad + z^3 x \ldots \quad + z^3 y \ldots .
\end{aligned}$$

$$+y^2z^2$$

$$\left[-a_2bc_1c_2+a_2b_1c_2^2-a_3b_1b_2c+a_3b_2^2c_1-b_1^2c_1^2\right.$$
$$+\left\{-a_2b_1c+a_2b_3c_1-a_3bc_1+a_3b_1c_2\right\}f+(b_1c+b_2c_1)\,b_1g+(bc_1+b_1c_2)\,c_1h$$
$$-b_2^2g^2-c_2^2h^2-bcgh+bg^2f+ch^2f+f^2gh$$
$$+\left\{2b_1c_1f+bc_2g+b_2ch-c_2hf-b_2fg\right\}l$$
$$+\left\{a_2bc-a_2b_2c_2-(bc_1+b_1c_2)\,g-2c_1fh\right\}m$$
$$+\left\{a_2bc-a_2b_2c_2-(b_1c+b_2c_1)\,h-2b_1fg\right\}n$$
$$-\left\{bc+f^2\right\}l^2-\left\{ch+fg\right\}m^2-\left\{bg+hf\right\}n^2+4\left\{b_2g+c_2h\right\}mn$$
$$+\left\{bc_1-2b_1c_2\right\}nl+\left\{b_1c-2b_2c_1\right\}lm$$
$$+2c_1m^2+2b_1n^2-c_2m^2l+b_2n^2l+4flmn$$
$$\left.-3m^2n^2\right]$$

$$+z^2x^2\ldots\quad+x^2y^2\ldots.$$

$$+x^2yz$$

$$\left[a_2^2b_2c-a_2a_3bc+a_3^2bc_2-\left\{2ab_1c_1+a_2^2c_2+a_3^2b_3\right\}f\right.$$
$$-\left\{abc_2-a_2bc_1+a_2b_1c_2\right\}g-\left\{ab_2c-a_2b_1c+a_2b_2c_1\right\}h$$
$$+a_2a_3f^2-b_1c_1gh+(ac_2+2a_2c_1)\,hf+(ab_2+2a_2b_1)\,fg+2b_2g^2h+2c_2h^2g$$
$$+\left\{abc-a_2b_1c-a_2b_2c_1-a_2bc_1-a_3b_1c_2-of^2-4fgh\right\}l$$
$$+\left\{2ab_2c_1+2a_2b_1c_1-2a_2b_2g-a_2c_2h-2b_1g^2+a_2fg\right\}m$$
$$+\left\{2ab_1c_2+2a_2b_1c_1-a_2b_2g-2a_2c_2h-2c_1h^2+a_2fh\right\}n$$
$$-\left\{b_1c_1+2b_2g+2c_2h\right\}l^2-2\left\{ac_2+2a_2c_1\right\}m^2-2\left\{ab_2+2a_2b_1\right\}n^2$$
$$+\left\{2af-3gh\right\}mn+\left\{5a_2b_2-3a_2f+4b_1g\right\}nl+\left\{5a_2c_2-3a_2f+4c_1h\right\}lm$$
$$+4fl^2+4gm^2l+2a_2m^2n+4hn^2l+2a_2n^2m$$
$$\left.-7l^2mn\right]$$

$$+y^2zx\ldots\quad+z^2xy\ldots.$$

Some Properties of Co-normal Points on a Parabola.

By R. Tucker.

[*Read June 12th*, 1890.]

1. If the coordinates of a point on a parabola,

$$y^2-4ax=0,$$

be $(am^2, 2am)$, then the equations to the tangent and normal at the point are

$$x-my+am^2=0 \quad \text{.....(i.),}$$

and

$$mx+y-a(m^2+2m)=0 \quad \text{.....(ii.),}$$

and to the chord through (m), (m'), is

$$y(m+m')-2x-2amm'=0 \quad \text{.....(iii.).}$$

If we write (ii.) in the form

$$am^3+(2a-x)m-y=0 \quad \text{.....(iv.),}$$

we see that from a given point (x, y), we can draw three normals to the curve, with the condition

$$\Sigma m=0.$$

Let O be the point, and $P(m_1)$, $Q(m_2)$, $R(m_3)$ the corresponding points on the parabola; then I call these latter *co-normal* points, and the circle through them a *co-normal* circle.

2. Since
$$S_1 \equiv \Sigma m=0,$$
$$S_2 \equiv \Sigma m^2=-2(m_1m_2+\ldots+\ldots)=-2\Sigma m_1m_2,$$
$$S_3=3m_1m_2m_3=3\mu,$$
$$S_4=S_2^2/2;$$

also
$$m_1^2-m_2m_3=\ldots=\ldots=m_1^2+m_1m_2+m_3^2=S_2/2.$$

3. In the case when P, Q, R are *any* three points on the curve the circle PQR is

$$x^2+y^2-ax\left[S_2+\Sigma m_1m_2+4\right]+ay\left[S_1.\Sigma m_1m_2-\mu\right]/2-a^2\mu S_1=0 \ldots\text{(i.),}$$

and the tangent-circle pqr is

$$x^2+y^2-ax\left[1+\Sigma m_1m_2\right]-ay\left[S_1-\mu\right]+a^2\Sigma m_1m_2=0\ldots\ldots\text{(ii.)}.$$

If the points are co-normal points, then these equations take the form

$$x^2+y^2-ax\,(S_2+8)/2-ay\mu/2=0\ \ldots\ldots\ldots\ldots\text{(iii.)},$$

$$x^2+y^2+ax\,(S_2-2)/2+ay\mu-a^2S_3/2=0\ \ldots\ldots\ldots\ldots\text{(iv.)}.$$

4. The coordinates of p (§ 3), for co-normal points, are

$$(am_2m_3,\ -am_1);$$

the coordinates of P', mid-point of QR, are

$$\left[a\,(m_2^2+m_3^2)/2,\ -am_1\right];$$

and the coordinates of p', mid-point of qr, are

$$(-am_1^2/2,\ am_1/2).$$

5. The equation to $P'Q'R'$ (N.P. circle of PQR) is

$$x^2+y^2-ax\,(3S_2-8)/4+ay\mu/4+a^2\,(S_4-4S_2)/4=0\ \ldots\ldots\text{(i.)},$$

and to $p'q'r'$ (N.P. circle of pqr), is

$$x^2+y^2+ax\,(S_2+2)/4-ay\mu/2=0\ \ldots\ldots\ldots\ldots\text{(ii.)};$$

this circle evidently passes through the vertex.

6. The circle through the vertex, which touches the parabola at (m_1), is

$$x^2+y^2-a\left[3m_1^2+4\right]x+am_1^3y=0\ \ldots\ldots\ldots\ldots\text{(i.)},$$

if the parameter of the fourth point of section (P_1) be M, then, because

$$0+2m_1+M=0,$$

we see that the points P_1, $(Q_1,\ R_1)$ are co-normal points.

7. The centre of curvature at P is

$$\left[a\,(3m_1^2+2),\ -2am_1^3\right],$$

therefore the circle of curvature at P is

$$x^2+y^2-2ax\,(3m_1^2+2)+4am_1^3y-3a^2m_1^4=0\ \ldots\ldots\ldots\text{(i.)};$$

and, because

$$3m_1+M'=0,$$

the fourth points of section P_2, $(Q_2,\ R_2)$ are co-normal points.

The equation to circle $P_2Q_2R_2$ is

$$x^2+y^2-ax\,(9S_2+8)/2+27ay\mu/2=0 \ldots\ldots\ldots\ldots\ldots\text{(ii.)};$$

the radical axis of this and PQR is

$$2S_2x=7\mu y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}.$$

The tangent-circle $(p_3q_2r_3)$ is

$$x^2+y^2+ax\,(9S_2-2)/2-27ay\mu-9a^2S_2/2=0\ldots\ldots\ldots\text{(iv.)}.$$

The radical axis of this and pqr is

$$S_2x-7\mu y-aS_2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)}.$$

8. The equation to $P'p'$ is

$$yS_2+3m_1x+am_1\,(m_1^2+2m_2m_3)/2=0,$$

whence we see that

$$P'p',\ Q'q',\ R'r' \text{ cointersect in } (-aS_2/12,\ -3a\mu/2S_2).$$

9. From § 3 (iii.), § 5 (ii.), we see that the radical axis of PQR, $p'q'r'$ is the tangent at the vertex, and, from § 7, that the radical centre of the three circles of curvature is $(-aS_2/4,\ 3a\mu/2S_2)$.

10. The equation to QR is

$$2x+m_1y+2am_2m_3=0,$$

and the equation to qr is $\quad x-m_1y+am_1^2=0;$

hence these cut in $\left[a\,(m_1^2+2m_2m_3)/3,\ aS_2/3m_1\right]$, *i.e.*, in p_3 say; hence the centroid of all such triangles as $p_3q_3r_3$ lies on the tangent at the vertex.

11. The line Pp is

$$yS_2-6m_1x+a\,(6\mu+m_1S_2)=0\ ;$$

hence Pp, Qq, Rr cointersect in $(aS_2/6,\ -6a\mu/S_2)$.

12. The perpendicular from q on RP is

$$m_2x-2y=a\,(2m_2+\mu)\ ;$$

hence the perpendiculars from p on QR, q on RP, r on PQ, meet in $(2a,\ -a\mu/2)$.

13. The orthocentre of PQR is

$$\left[a\,(S_2-8)/2,\ -a\mu/2\right];$$

the orthocentre of pqr is $(-a, a\mu)$;

and the orthocentre of PQr (π_r) is

$$\left[-a(2+m_1m_2),\ -a(2m_3+\mu)\right];$$

hence $A\pi_r$ is parallel to the tangent at R,

$A\pi_p$ „ „ „ P,

and $A\pi_q$ „ „ „ Q.

The centroid of $\pi_p\pi_q\pi_r$ is

$$\left[a(S_2-12)/6,\ -a\mu\right].$$

14. The centroid of PQR is $(aS_2/3, 0)$,

the centroid of pqr is $(-aS_2/6, 0)$,

the centroid of PQr (g_3) is $(aS_2/6, -am_3)$;

hence g_1, g_2, g_3 are collinear, and their centroid is

$$(aS_2/6,\ 0).$$

15. From § 1, the coordinates of O are

$$\left[a(S_3+4)/2,\ a\mu\right].$$

The isoclinal (through P) to PN with the normal is

$$y-m_1x+a(m_1^3-2m_1)=0 \quad \text{.....(i.)};$$

hence the similar lines through Q, R meet it in

$$O',\ \left[a(S_2-4)/2,\ -a\mu\right].$$

The equation to OO' is

$$4y-2\mu x+a\mu S_2=0 \quad \text{.....(ii.)}.$$

16. Since PO cuts the axis at

$$\left[a(m_1^2+2),\ 0\right],$$

the perpendicular to PO, through this point, is

$$m_1y-x+a(m_1^2+2)=0 \quad \text{.....(i.)};$$

this meets the similar line for Q in r_5, say

$$\left[a(2-m_1m_2),\ am_3\right].$$

The circle $p_5q_5r_5$ is

$$x^2+y^2-ax\left[S_2+6\right]/2-a\mu y+a^2(S_2+4)/2=0 \quad \text{.....(ii.)},$$

of which the diameter is SO. By its construction the triangle $p_3q_3r_3$ has its sides parallel to the sides of pqr, and, as the radii of the circles are equal, therefore the triangles are equal. The circles touch at S. (§ 16 furnishes an easy way of finding O when P, Q, R are given.) S is the centre of perspective of the triangles. Since, with the usual notation, $ST = SG$, therefore PO cuts the circle in the mid-point of PG, and so for the other normals.

17. Perpendiculars drawn to the tangents at P, Q, R, at the points where they meet the axis, since their equations are

$$y + m_1 x + a m_1^3 = 0, \ \ldots,$$

meet in Π, $\quad \left[-aS_3/2, \ -a\mu\right]$.

This point is the extremity of the diameter of pqr through S. The polar of Π, with respect to the parabola, is

$$2x + \mu y = aS_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)};$$

hence it is perpendicular to OO', and meets that line in

$$\left[aS_3/2, \ 0\right],$$

i.e., on the axis.

18. The circle round $P\,(m_1)$, $P_1\,(-2m_1)$, $P_2\,(-3m_1)$ cuts the curve again (because $m_1 - 2m_1 - 3m_1 + 4m_1 = 0$) in $P_3\,(4m_1)$; hence P_3, (Q_3, R_3) are co-normal points.

The equation to PP_1P_2 is

$$x^3 + y^3 - ax\,(15m_1^2 + 4) - 5am_1^3 y + 24a^3 m_1^4 = 0 \ldots\ldots\ldots\ldots\text{(i.)},$$

and the radical centre of it and the circles for Q, R is

$$\left[4aS_3/5, \ 48a\mu/5S_3\right].$$

19. Through P, Q, R draw parallels to

(i.) the tangents at Q, R, P;

(ii.) „ „ R, P, Q;

and let P_q, Q_r, R_p; P_r, Q_p, R_q be the points where the sets (i.), (ii.) respectively meet the parabola; their parameters are

$$(2m_3 - m_1, \ 2m_3 - m_2, \ 2m_1 - m_3), \quad (2m_3 - m_1, \ 2m_1 - m_2, \ 2m_2 - m_3);$$

hence the two sets are co-normal points.

The mid-point of P_qP_r, &c. is on the diameter through P_3, &c.

20. The circle PQP_q, because

$$m_1+m_2+(2m_3-m_1)-3m_2=0,$$

passes through Q_3; therefore QRQ_r, RPR_p pass through R_3, P_3, respectively.

Similarly, PRP_r, QPQ_p, RQR_q pass through R_2, P_2, Q_3, respectively.

21. Again, because

$$m_1+(2m_2-m_1)-2m_2+0=0,$$

therefore

$$\left.\begin{array}{llll} PP_qA & \text{passes through} & Q_1 \\ QQ_rA & ,, & ,, & R_1, \\ RR_pA & ,, & ,, & P_1, \end{array}\right\}$$

and

$$\left.\begin{array}{llll} PP_rA & \text{passes through} & R_1, \\ QQ_pA & ,, & ,, & P_1, \\ RR_qA & ,, & ,, & Q_1. \end{array}\right\}$$

22. The circles $P_qQ_rR_p$, $P_rQ_pR_q$ are given by

$$x^2+y^2-ax\,(7S_2+8)/2-a\nu y/2=0 \quad \ldots\ldots\ldots\ldots\text{(i.)},$$

$$x^2+y^2-ax\,(7S_2+8)/2-a\nu' y/2=0 \quad \ldots\ldots\ldots\ldots\text{(ii.)},$$

where

$$\nu \equiv (2m_1-m_3\,.\,2m_2-m_1\,.\,2m_3-m_2),$$

$$\nu' \equiv (2m_1-m_2\,.\,2m_2-m_3\,.\,2m_3-m_1),$$

and

$$\nu+\nu'=20\mu.$$

Hence (i.), (ii.), intersect on the axis, and abscissæ of their centres are the same.

23.
$$\Delta P_qQ_rR_p+\Delta P_rQ_pR_q=162\mu a^2.$$

24. PP_q, QQ_p meet in R_r,

$$\left[a\,(S_2-4m_1m_2)/2,\ -am_3\right],$$

i.e., on the diameter through r; and the centroid of P_p, Q_q, R_r is

$$\left[a\,(5S_2/6),\ 0\right].$$

25. The circle through mid-points of P_qP_r, Q_rQ_r, R_pR is

$$x^2+y^2-ax\,(9S_2+32)/2+a\mu y/4+a^2\,(10S_3+32S_2)=0.$$

26. The centroids of $P_qQ_rR_p$, $P_rQ_pR_q$ coincide in

$$\left[7aS_2/3,\ 0\right].$$

Their orthocentres are $\left[a\,(7S_2-8)/2,\ -a\nu/2\right]$

and $\left[a\,(7S_2-8)/2,\ -a\nu'/2\right]$.

27. The tangents at Q_r, R_p meet in π_q,

$$\left[a\,(2m_3-m_2)(2m_1-m_3),\quad -a\,(2m_3-m_1)\right],$$

and at Q_p, R_q in π'_q,

$$\left[a\,(2m_1-m_2)(2m_2-m_3),\quad -a\,(2m_3-m_1)\right];$$

hence the centroids of $\pi_p\pi_q\pi_r$, $\pi'_p\pi'_q\pi'_r$ coincide in

$$\left[-7aS_2/6,\ 0\right].$$

28. Let PS, QS, RS meet the curve again in t_1, t_2, t_3; then, since t_1 is given by $\left[a/m_1^2,\quad -2a/m_1\right]$,

the circle $t_1t_2t_3$ is

$$x^2+y^2-a\left[\Sigma 1/m^2+4\right]x+ay/2\mu-a^2\Sigma 1/m/\mu=0\ldots\ldots\ldots\ldots\text{(i.)}.$$

The tangents at t_1, t_2, t_3, meet in t'_1, t'_2, t'_3, where t'_1 is the point

$$\left[am_1/\mu,\quad am_1^2/\mu\right];$$

hence the circle $t'_1t'_2t'_3$ is

$$x^2+y^2-ax-ay\,(S_2+2)/2\mu=0\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

which passes through the vertex as well as the focus.

29. In fact, if the tangents at the co-normal points P, Q, R cut the line given by $x=ka$, and from the points of section the second set of tangents be drawn, touching the curve at T_1, T_2, T_3; then, since T_1 is given by (k/m_1), $\Sigma m_1m_2=0$; hence, by § 3 (ii.), the tangent-circle for $T_1T_2T_3$ is

$$x^2+y^2-ax+ay\,(k\,.\,S_2+2k^3)/2\mu=0\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}.$$

30. The "N.P." circle of $t'_1t'_2t'_3$ is

$$x^2+y^2+ax/2+ay\,(2-3S_2)/4\mu+a^2\,(S_4-S_2)/2\mu^2=0\ldots\ldots\ldots\text{(i.)}.$$

The centroid of the triangle is

$$\left[0,\ aS_2/3\mu\right].$$

31. The following method of obtaining sets of co-normal points is of some interest:—

Draw PS and produce it to meet the curve in L; join LA and draw Aa_1 perpendicular to AL. The parameter of a_1 is $4m_1$ (and it is therefore the point P_3 of § 17); similarly, from a_1 we can get a_2, and so on. Let b_1, c_1 be the corresponding points for Q_1, R_1; then (a_1, b_1, c_1), (a_2, b_2, c_2), ... are co-normal sets. Again, reversing the above process, *i.e.*, join PA, draw AL' perpendicular to AP, then join L' to S and produce to meet the curve in a_1', ..., and we have sets

$$(a_1', b_1', c_1'), \quad (a_2', b_2', c_2'), \ \dots,$$

with parameters $m_1/4$, $m_1/16$, &c. a_1a_1', a_2a_2', &c. meet the axis where it is cut by the tangent at P, &c., and the tangents at a_1, a_1', meet on the ordinate of P, and so on.

32. The circle Pa_1a_1' is

$$x^2+y^2-\left(4+\frac{357}{16}m_1^2\right)ax+\frac{425}{32}am_1^3y-\frac{21}{4}a\,m_1^4=0 \ \dots\dots \text{(i.)},$$

and it cuts the curve in a fourth point $(-21m_1/4)$; hence this point and the analogous ones form a co-normal system.

33. The equations to $a_1b_1c_1$, $a_1'b_1'c_1'$ are

$$x^2+y^2-ax\,(8S_2+4)-32a\mu y=0 \dots\dots\dots\dots\dots\dots \text{(i.)},$$

and

$$x^2+y^2-ax\,(S_2+128)/32-ay\mu/128=0 \dots\dots\dots\dots\dots \text{(ii.)}.$$

34. The sum of the squares of the tangents, taken once, from a_1, b_1, c_1 to pqr, PQR, are respectively

$$a^2\,(272S_4+93S_2/2) \quad \text{and} \quad 240S_4a^2.$$

35. Since the parameters of P_q, P_r, a_1 are $2m_2-m_1$, $2m_3-m_1$, and $4m_1$, their sum $=0$; hence the three points form a co-normal system, as do Q_r, Q_p, b_1; R_p, R_q, c_1. (*Cf.* § 17.)

36. The equation to any parabola passing through the co-normal system P, Q, R is

$$(y+\lambda x)^2=ax\left[8+\lambda^2S_2\right]/2+\lambda ay\left[2S_2+\mu\lambda\right]/2+4\lambda\mu a^2 \dots\dots \text{(i.)},$$

where λ is arbitrary.

If we make the curve pass through O, we find $\lambda^2=2$, *i.e.*, the two parabolas are

$$(y\pm\sqrt{2}\,x)^2=ax\,(S_2+4)+ay\,(\mu\pm\sqrt{2}\,S_2)\pm4\sqrt{2}\,\mu a^2 \dots\dots \text{(ii.)}.$$

This latter may be written in the form

$$y^2+2x^2-ax(S_2+4)-ay\mu = \pm\sqrt{2}\left[ayS_2+4\mu a^2-2xy\right] \ldots\ldots \text{(iii.)}$$

Hence we see that the ellipse through the four points and through the vertex is given by

$$y^2+2x^2-ax(S_2+4)-ay\mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)},$$

and the rectangular hyperbola through the four points is

$$2xy = ayS_2+4\mu a^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(v.)}$$

37. From (ii.) of § 36, we see that the mid-point of each of the intercepts made by the parabolas on the axis of the given parabola is the projection of O on that axis.

38. From § 15 and § 36 (iv.), we see that the centre of the ellipse is at the mid-point of AO.

39. The centre of the hyperbola is $(aS_2/2,\ 0)$; *cf.* § 17. The tangent at O is

$$2y+x\mu = a\mu(S_2+8)/2,$$

which shows that it coincides with the perpendicular through O on the polar of O with regard to the parabola.*

40. The second parabola, which passes through A and the co-normal points, is given by

$$2x^2 = axS_2+a\mu y.$$

41. $SP = a(1+m_1^2)$, ... , ... ;

$$\therefore\quad SP.SQ.SR = a^3\left[1+S_2+\frac{S_2^2}{4}+\mu^2\right] = a.SO^2.$$

42. Many results may be got by taking μ = constant; thus (§ 9) in this case the locus of the radical centres of the three circles of curvature at P, Q, R is the rectangular hyperbola

$$8xy = -3a^2\mu.$$

[43. We refer to *Milne and Davis*, the Parabola, pp. 37 and 56, for geometrical proofs of §§ 1 and 3 (iii.). From this last it is obvious that, given any two points on the curve, the co-normal system can be determined by finding the 4th point of section of a circle through them and the vertex with the curve.

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* *Cf.* also § 15 (ii.).

44. A referee states that § 5, (ii.), was set as a problem in the June 1889 Examination at Caius College, Cambridge; and I find a long proof of § 6, (i.), in the *Nouvelles Annales des Mathématiques*, March, 1890, by M. Lemaire.

45. *Milne*, p. 34, illustrates § 16.

46. On §§ 19–21, a referee remarks that the result may be generalized thus: Let PQR be one set of co-normal points, and $P'Q'R'$ another set; through P, Q, R, draw parallels to the tangents at P', Q', R' meeting the curve again in L, M, N, respectively, then L, M, N are also co-normal points. Again, the circle $PP'L$ meets the curve in P'_3, and the circle PLA meets the curve in P'_2, P'_2 being (-2μ), $P'_3(-3\mu)$, where μ is the parameter of P'.

47. If p_t, q_t, r_t are the points of contact of tangents parallel to QR, RP, PQ, then these are given by

$$(p_t)\ \left[am_1^2/4,\ \ -am_1\right],\ \ldots,\ \ldots.$$

The tangent circle for p_t, q_t, r_t is

$$x^2+y^2+ax\left[S_2-8\right]/8-a\mu y/8-a^2S_2/8=0.$$

The perpendicular from r_t on PQ is

$$m_3x+2y=2am_3+am_3^3/4;$$

hence the three like perpendiculars meet in

$$a\left[2+S_2/8\right],\quad -a\mu/8.$$

The orthocentre of $p_tq_tr_t$ is

$$a\left[-4+S_2/8\right],\quad a\mu/16.$$

48. Since the co-normal points $(P,\ Q,\ R)$ are on a circle through the vertex, the inverse points, with respect to A, which lie upon a cissoid, are on a straight line. Its equation is

$$\mu y+(S_2+8)\,x=2k^2/a,$$

where k is the modulus of inversion.

49. If

$$\mu_1\equiv p_1m_1+q_1m_2+r_1m_3,$$
$$\mu_2\equiv p_2m_1+q_2m_2+r_2m_3,$$
$$\mu_3\equiv p_3m_1+q_3m_2+r_3m_3,$$

then μ_1, μ_2, μ_3 will be a co-normal system, provided

$$\Sigma p=\Sigma q=\Sigma r.\Big]$$

The following presents have been received during the recess :—

Cabinet likeness of Mr. A. B. Basset, for the Society's Album.

"Proceedings of the Royal Society," No. 289–294.

"Educational Times," for July, August, September, and October, 1890.

"Proceedings of the Royal Society of Edinburgh," Vols. xv. and xvi.

"Proceedings of the Royal Irish Academy," 3rd Ser., Vol. i., No. 3; June, 1890.

"Scientific Proceedings of the Royal Dublin Society," Vol. vi., Parts 7, 8, 9.

"The Scientific Papers of James Clerk-Maxwell," edited by W. D. Niven, M.A., F.R.S., 2 Vols., 4to; Cambridge, 1890.

"Proceedings of the Canadian Institute, Toronto," Third Series, Vol. vii., Fasc. No. 2; 1890, April.

"Smithsonian Report, 1887," 8vo; Washington, 1889.

"Memoirs of the National Academy of Sciences," Vol. iv., Part 2, 4to; Washington, 1890.

"Catalogue of Stars observed at the United States Naval Observatory during the years 1845 to 1877," Third Edition, 4to; Washington, 1889.

"Report of the Superintendent of the U.S. Naval Observatory for the year ending June 30, 1889," 8vo; Washington, 1889.

"Bulletin des Sciences Mathématiques," Juin–Sept., 1890.

"Annales de la Faculté des Sciences de Toulouse," Tome iv., Année 1890, 4to, Paris, 1890; 1er fascieule.

"Annales de l'École Polytechnique de Delft," Tome v., 1890, 3me and 4me Livr., 4to; Leide, 1890.

"Atti della Reale Accademia dei Lincei—Memorie," Vol. v.; Roma, 1888.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. vi., Fasc. 8–10, 12 (1° Semestre); Fasc. 1–3 (2° Semestre).

"American Journal of Mathematics," xii., 4.

"Annali di Matematica," Serie ii., Tomo xviii., Fasc. 2.

"Journal für die reine und angewandte Mathematik," Band cvi., Heft 3 u. 4; Band cvii., Heft 1.

W. Pfeffer.—(i.) "Über Aufnahme und Ausgabe Ungelöster Körper," (ii.) "Zur Kentniss der Plasmahaut und der Vacuolen," &c., 8vo; Leipzig, 1890.

Paul Starke.—"Arbeitsleistung und Wärme-entwickelung bei der verzögerten Muskelzuckung," 8vo; Leipzig, 1890.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," i., 8vo; Leipzig, 1890.

"Jornal de Sciencias Mathematicas e Astronomicas." Vol. ix., No. 5.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," i.–xix.

"Memorias de la Sociedad Científica—Antonio Alzate," Tomo iii., Núms. 9 y 10; Mexico, 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band xiv., Stücke 6–8.

"Rendiconti del Circolo Matematico di Palermo," Tomo iv., Fasc. v. (Sett-Ott. 1890.)

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxiv., Livr. 2 and 3.

"Jahrbuch über die Fortschritte der Mathematik," Band xix., Heft 3; Berlin, 1890.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang, xxxv., Heft 1.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," 1890, Nos. 107–114; Index to ditto, pp. 1–80.

"Nyt Tidsskrift for Mathematik," A. Forste Aargang, Nos. 2, 3; B. Forste Aargang, No. 2.

"Calendar of the Imperial University of Japan for the years 1889–90, 8vo; Tōkyō, 1889.

APPENDIX.

The following notes, received from Mr. J. Griffiths* (December, 1889), are of interest in connexion with the papers on "Isoscelians" (*Proc. Lond. Math. Soc.*, Vol. XIX.), and on "Isoscelian Hexagrams" in this Volume (p. 4).

I. Using the Fig. *A* (*Proc.*, Vol. XIX., p. 166), and calling the angles

$$\theta, \phi, \psi, \quad [\theta+\phi+\psi = 180^\circ] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

in place of A, B, C, he gives the trilinear coordinates of P by the equations

$$\alpha \sin\theta \sin(\theta+A) = \beta \sin\phi \sin(\phi+B) = \gamma \sin\psi \sin(\psi+C) \ldots \text{(ii.)}.$$

These relations are unaltered when

$$\pi-\theta-A, \quad \pi-\phi-B, \quad \pi-\psi-C$$

are written for θ, ϕ, ψ respectively; hence we arrive at the same point with the negative θ-, ϕ-, ψ-lines.

(1) When $\theta = A$ or $\pi-2A$, $\phi = B$ or $\pi-2B$, $\psi = C$ or $\pi-2C$,

we arrive at the Isoscelian results (*loc. cit.*, p. 166).

(2) When $\theta = \frac{\pi}{2} - \frac{A}{2}$, $\phi = \frac{\pi}{2} - \frac{B}{2}$, $\psi = \frac{\pi}{2} - \frac{C}{2}$,

the positive and negative circles coincide: the coordinates of P, in

* Mr. Griffiths was a referee for the paper on "I. Hexagrams," and has kindly put the following notes, amongst others, at our service.

this case, are proportional to

$$\sec^2\frac{A}{2},\ \sec^2\frac{B}{2},\ \sec^2\frac{C}{2}\ *,$$

and the common equation of the circles is $I = K.L^2$, where

$$L \equiv \Sigma a \sin A,$$

$$I \equiv \Sigma \cos^4\frac{A}{2}.a^2 - 2\Sigma \cos^2\frac{B}{2}\cos^2\frac{C}{2}\beta\gamma,$$

and

$$K \equiv \cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\Big/4\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2.$$

Conversely, if the point $P(\alpha, \beta, \gamma)$ is known, θ, ϕ, ψ have to be determined from (i.), (ii.), *i.e.*, from

$$\cos(2\theta + A) = \cos A - \frac{x}{a}, \ldots, \ldots;$$

then either $x = 0$, or

$$\frac{x^2}{a\beta\gamma} - \tfrac{1}{2}\Sigma\left(\frac{1}{a^2} + \frac{2\cos A}{\beta\gamma}\right)x + \Sigma\frac{\sin B \sin C}{a} = 0.$$

II. Again, if DEF is a triangle inscribed in the circle ABC, so that its sides are θ-, ϕ-, ψ- lines for A, B, C, with the condition (i.), the isogonal point of the vertex $D(\alpha, \beta, \gamma)$ lies on the perpendicular from B on EF. Such a triangle [as DEF (*l.c.*, p. 167)] Mr. Griffiths calls a perpendicular isogonal triangle.

III. If FOF_1 is a diameter, then Mr. Griffiths calls DEF_1 a parallel isogonal triangle—*i.e.*, a triangle such that the isogonal of each vertex lies on a parallel to the opposite side.

IV. If we take such a triangle (as DEF_1), then its sides, together with those of ABC, are six tangents to a parabola.

V. The pedal lines with reference to ABC of the vertices of an in-parallel isogonal triangle (III.) cointersect on the directrix of the six-tangents parabola (IV.).

These results (IV. and V.) are particular cases of more general propositions in Mr. Griffiths' "Notes on the Geometry of the Plane Triangle" (1887). He obtains the equation to the positive circle

* This is the "Gergonne" point of the triangle ABC, and the lines are *direct Isoscelians*. (February, 1891.)

(Vol. XIX., p. 167) in the following form :—

If $\rho_1 = \sin\theta \sin(\theta+A)$, $\rho_2 = \sin\phi \sin(\phi+B)$,

$\rho_3 = \sin\psi \sin(\psi+C)$, $\rho = \Sigma(\sin A \rho_2\rho_3)$,

$S = \Sigma(\sin A\beta\gamma)$, $L = \Sigma(\sin A\alpha)$,

where $\theta+\phi+\psi = A+B+C = \pi$,

the equation is

$$\rho^2 S = \sin A \sin B \sin C \,.\, L(\lambda\alpha+\mu\beta+\nu\gamma),$$

where

$$\lambda \sin A = \left[\sin\theta\rho_3 + \sin(\theta+A)\rho_2\right]\rho_1^2 \sin(C-\phi+\psi),$$

$$\mu \sin B = \left[\sin\phi\rho_1 + \sin(\phi+B)\rho_3\right]\rho_2^2 \sin(A-\psi+\theta),$$

$$\nu \sin C = \&c.$$

For the equation to the negative circle, we must write in the above

$$\pi-\theta-A,\ \pi-\phi-B,\ \pi-\psi-C \text{ for } \theta, \phi, \psi.$$

The following results are closely connected with the subject of "Isoscelians."

If through A we draw the lines AD, AD', cutting BC, of the triangle ABC, in the points D, D', so as to make with AB, AC, respectively, the angle $90°-A$, and so for the other angular points, we get two sets of lines, which we propose to call positive and negative *complemental* lines. These lines are perpendicular to the positive and negative oblique Isoscelian lines, hence the triangles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, which they form, are similar to the Isoscelian triangles. The coordinates are—

of γ_1, $\alpha/\cos A \cos B = \beta/\cos 2A \cos 2B = \gamma/-\cos A \cos 2B$;

of γ_2, $\alpha/\cos 2A \cos 2B = \beta/\cos A \cos B = \gamma/-\cos 2A \cos B$.

The angles are

$$\alpha_1(2C-B),\quad \beta_1(2A-C),\quad \gamma_1(2B-A);$$

$$\alpha_2(2B-C),\quad \beta_2(2C-A),\quad \gamma_2(2A-B);$$

and the sides, $\beta_1\gamma_1 = 2\rho_1 \sin(2C-B)$,

$$\rho_1 \equiv 2R\,\frac{(\cos A \cos B \cos C + \cos 2A \cos 2B \cos 2C)}{\sin\alpha_1 \sin\beta_1 \sin\gamma_1};$$

$$\beta_2\gamma_2 = 2\rho_2 \sin(2B-C),$$

$$\rho_2 \equiv 2R\,\frac{(\ldots+\ldots)}{\sin\alpha_2 \sin\beta_2 \sin\gamma_2}.$$

The equations to the circles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ are

$$\alpha\beta\gamma+\ldots+\ldots+\frac{a\alpha+\ldots+\ldots}{\sin a_1 \sin\beta_1 \sin\gamma_1}\left[\cos C\cos 2B\sin a_1\alpha+\ldots+\ldots\right],$$

$$\alpha\beta\gamma+\ldots+\ldots+\frac{a\alpha+\ldots+\ldots}{\sin a_2 \sin\beta_2 \sin\gamma_2}\left[\cos B\cos 2C\sin a_2\alpha+\ldots+\ldots\right].$$

The in-complemental triangles $\alpha_3\beta_3\gamma_3$, $\alpha_4\beta_4\gamma_4$ have their angles

$$a_3(=a_1),\ \beta_3(=\beta_1),\ \gamma_3(=\gamma_1)\ \text{on}\ CA,\ AB,\ BC;$$

and $\quad a_4(=a_2),\ \beta_4(=\beta_2),\ \gamma_4(=\gamma_2)\ \text{on}\ AB,\ BC,\ CA;$

and the sides $\quad \beta_3\gamma_3=2\rho_3\sin a_3,\quad \beta_4\gamma_4=2\rho_4\sin a_4,$

where $\quad \rho_3=2R\sin A\sin B\sin C/D=\rho_4,$

and $\quad D\equiv\sin 2A\sin 2B+\sin 2B\sin 2C+\sin 2C\sin 2A$

$$=\tfrac{1}{2}\left[\cos 2(B-C)+\cos 2(C-A)+\cos 2(A-B)\right.$$
$$\left.-(\cos 2A+\cos 2B+\cos 2C)\right]$$
$$=2\left[\cos(A-B)\cos(B-C)\cos(C-A)+\cos A\cos B\cos C\right].$$

The equations to the circles are

$$D^2(\alpha\beta\gamma+\ldots+\ldots)$$
$$=(a\alpha+\ldots+\ldots)\left[\sin 2B\sin 2C(\sin 2C+\sin 2A)\sin\gamma_3\alpha+\ldots+\ldots\right],$$

and
$$D^2(\alpha\beta\gamma+\ldots+\ldots)$$
$$=(a\alpha+\ldots+\ldots)\left[\sin 2B\sin 2C(\sin 2A+\sin 2B)\sin\beta_4\alpha+\ldots+\ldots\right].$$

Mr. Kempe sends us the following Note:—

"In the *Quarterly Journal of Mathematics* of June last, No. 96, p. 332, Mr. P. J. Heawood, of Durham University, in an article 'On the Map-colour Theorem,' points out that there is a flaw in the proof given by me in the *American Journal*, Vol. II., p. 193, and *Nature*, Vol. XXI., p. 399, of which a short account appeared in these *Proceedings*, Vol. X., p. 229. My proof consisted of a method by which any map could be coloured with four colours. Mr. Heawood gives a case in which the method fails, and thus shows the proof to be erroneous. I have not succeeded in remedying the defect, though it can be shown that the map which Mr. Heawood gives can be coloured with four colours, and thus his criticism applies to my proof only and not to the theorem itself. It may be as well to point out that the statement made by Mr. Heawood in the last few lines of his paper, that a map can be coloured with but three colours if all its divisions are touched by an even number of others, is erroneous, unless we introduce the qualification that the map is such that three divisions only meet at each point of concourse."

Dr. Glaisher sends the following correction to a statement contained in a note to his paper "On the Square of Euler's Series."

"In the note on p. 204 it is stated that the formula

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c. = \sigma(n)$$

only holds good when n is uneven, reference being made to Vol. XII., p. 170, of the *Messenger*, where the proof given applies only to uneven values of n. The theorem is, however, not restricted to uneven values, and is, in fact, two pages earlier in the same paper (p. 202), enunciated and proved in its general form. I can only account for the Note on p. 204, by supposing that I added it subsequently when § 34 of the paper had passed out of my mind."

Alexander John Ellis was born at Hoxton, in Middlesex, on the 14th of June, 1814. His father's name was Sharpe, but, by royal licence, this was changed in 1825 to Ellis. This step is said to have been taken in consequence of a bequest, made by a relative, "to enable him to devote his life to study and research, unhampered by pecuniary cares; and if this was so, he certainly carried out most faithfully his part of the bargain." He was sent first to Shrewsbury School, and was then placed at Eton, whence he proceeded to Cambridge, at which University, after having been elected to a Scholarship at Trinity College in 1835, he graduated B.A. in 1837.

The Tripos List was headed by three Johnians, our distinguished second President being Second Wrangler; then came another illustrious mathematician, George Green, of Caius, followed by five Trinity men, of whom D. F. Gregory was the leader, and Ellis next, as Sixth Wrangler. Nor was this all: he came out first in the second class of the Classical Tripos, Conybeare and Howson being in the first class. He never proceeded to the higher degree; but it was a source of gratification to himself, and to many of his friends, that at last his University recognised his merit by conferring on him, in June, 1890, the honorary degree of Doctor in Letters. Mr. Ellis entered the Middle Temple as a student, and remained a member thereof; he was never, however, called to the Bar. He was elected a Fellow of the Royal Society on the 2nd June, 1864, and served on the Council from 1880 to 1882; a Fellow of the Society of Antiquaries in 1870; President of the Philological Society from 1872 to 1874, and again from 1880 to 1882. He was also elected F.C.P.S. (1837), and F.C.P. (1873), and, in addition, was a member of many other scientific and learned societies, and was made a life-governor of University College, London (1886).

For the last few years his health had been infirm, especially since the loss of his wife in 1889. He died at his residence in Kensington,

on the evening of Tuesday, October 28th, 1890. The immediate cause of his death was the bursting of a blood-vessel.

Dr. Ellis was "a man of genial disposition, of very extended knowledge, and keen critical faculty."

Some persons have been disposed to consider him hypercritical and a man of many crotchets; but, notwithstanding, we have never heard any one say that he made himself disagreeable in putting forward his views. He won the respect of all. "His zealous, sincere, and unselfish devotion to the spread of knowledge" was recognised by all who came into contact with him.

His scientific interests were diversified, and included music as well as mathematics and philology. The chief subject of his labours was "Phonetics," or the science of pronunciation. Not long after leaving Cambridge, he busied himself, in conjunction with Mr. Isaac Pitman, in arranging a system of printing called by them Phonotypy, and in 1844 he published a description of this in his "Phonetics: a familiar exposition of the principles of that science."

In 1849 he was the proprietor, editor, and chief writer of the *Phonetic News*. In the course of his work upon this, and "Glossic"* and kindred matters, "he had occasion to look into the history of English pronunciation, and he was so fascinated by the novelty and extent of the subject, that he determined to devote himself to its study; and it occupied him, more or less, all the remainder of his life." The first part of his resultant work, which appeared in 1869, was entitled — "On Early English Pronunciation, with special reference to Shakspere and Chaucer, containing an investigation of the correspondence of writing with Speech in England, from the Anglo-Saxon period to the present day, &c." Part II. appeared in 1869; Part III., 1871; Part IV.. 1874; Part V., which was expected to appear in 1878, was not finished until 1889. The whole work contains about 2,500 pages, and was published jointly by the Philological, the Chaucer, and the Early English Text Societies.

Another subject in which Mr. Ellis took great interest was the Scientific Theory of Music, and in connexion with this we may mention that his work in this direction was mainly due to a study of Helmholtz's "Die Lehre von den Tonempfindungen." Of this treatise he had so high an opinion that in 1875 he published a translation of it, under the title of "The Sensations of Tone as a Physiological Basis for the Theory of Music," accompanied by an Appendix occupying 180 pages out of 824 (2nd edition, 1885).

* He published three lectures on this subject in 1870.

Other results of his studies were papers read before the Royal Society (1864, 1884), and the Society of Arts (1877, 1880). For his paper "On the Measurement and Settlement of Musical Pitch" (1877), this last Society awarded him a Silver Medal.

The first subject, however, which engaged Mr. Ellis's powers was that of Mathematics, and in an interesting autobiographical sketch in the third Appendix ("On the History of Stigmatic Geometry") to his pamphlet *Algebra Identified with Geometry* (1874), Mr. Ellis makes us acquainted with his early mathematical training :—

"About fifty years ago my father taught me Euclid after Playfair, and Algebra from the 'Elements of Algebra,' by Leonard Euler.* Although I fear I did not profit properly by such a book, it was something to have begun Algebra under the guidance of a mathematician like Euler. . . . About forty years ago I studied, as a freshman, Dean Peacock's Algebra. . . . But I owe most at this period to discussions with my late friend D. F. Gregory, from whom I derived the germ of the conception of *operation*, and not *quantity*, as the real meaning of algebraical expressions. About this time, also, I became acquainted with the works of Martin Ohm of Berlin."

He then enumerates several of Ohm's writings, one of which he translated and published under the title "The Spirit of Mathematical Analysis, and its Relation to a Logical System" (1843). He translated many others (out of nearly 30 volumes) though he did not publish them. He then dropped mathematics for a time and

"worked so hard at practical Phonology that in 1849 I was completely prostrated, and remained for some three years incapable of headwork. As I was recovering, I again recurred to my mathematical researches, and especially amused myself with De Morgan's *Trigonometry and Double Algebra*."

At last, at five o'clock on the morning of 20th March, 1853, he

"awoke suddenly with a conception of the application of Algebra as a measure of quantity,—the germ of the Algebra of proportion. This thought kept working in my brain for some months . . . until, on the 13th of August, I noted in my journal that I had 'very satisfactorily arranged the whole subject in my own mind.' . . . The next morning I awoke at six o'clock with the thought clear upon me, and I noted: 'I have made a real discovery in mathematics, and have discovered the real theory of Analytical Geometry, showing that Descartes's is only its simplest case.'"

Mr. Ellis then gives a detailed account of his discovery. We have made these extracts because Mr. Ellis looked upon "Stigmatics" as his chief mathematical work. Mr. Tristram Ellis informs us that his father has left a large quantity of MS. on the subject, "it being his belief that this branch of mathematics would become a very useful one."

Mr. Ellis was elected a member June 19th, 1865, and served on the Council during the Sessions 1866–7, 1867–8. He occasionally acted as a referee. In the early days of the Society he was a very regular attendant at our meetings, and frequently made short communications.

* Howlett's Edition, 1822.

The following only are recorded in our *Proceedings*:—

"On Angles and Parallels" (No. 11, p. 45).

"From a Pair of Conjugate diameters in an Ellipse, to find the Foci and Axes" (Vol. II., p. 250).

"Algebra considered as the Calculus of similar Triangles" (April 9th, 1874). This was a brief account of what is given at length in his pamphlet entitled "Algebra identified with Geometry," in a series of five Tracts:—

(1) Euclid's Conception of Ratio and Proportion.

(2) "Carnot's Principle" for Limits.

(3) The Laws of Tensors, or the Algebra of Proportion.

(4) The Laws of Clinants, or the Algebra of similar triangles lying upon the same plane.

(5) Stigmatic Geometry, or the Correspondence of Points in a Plane. A copy of this work (84 pp. 8vo) was presented to each member of this Society, and of the Association for the Improvement of Geometrical Teaching.

This last-named Association was formed in 1870. Mr. Ellis became a member, we believe, towards the close of 1872. He attended, for the first time, the annual meeting of January, 1873. From that date, until the meeting in January, 1886, he was a most regular attendant, and did yeoman's service. The pamphlet above referred to in a measure owes its publication to work done for the Association. This point is plainly brought out in the *Preliminary Notice*. Before we leave the subject of "Stigmatics," we quote the following statement of the author:—

"My ambition is to present the arsenal of mathematics with a new arm of precision, of which the following pages contain the specification, and a sketch of its action, power, and range."

Mr. Ellis also published:—

"Self-proving Examples in Arithmetic" (1855).

"Arithmetical Crutches for Limping Calculators and Gymnastics for Weak Ones" (1876).

"How to teach Proportion without reference to Commensurability, with additional notes on Collateral subjects" (1877).*

From the Royal Society's Catalogue we copy the following titles:

"On a more general Theory of Analytical Geometry, including the Cartesians as a particular case." [*Brit. Assoc.*, 1855, Part II., pp. 5–7.]

"On the Laws of Operation, and the Systematization of Mathematics." [*Royal Soc. Proc.*, Vol. X., 1859–60, pp. 85–94.]

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* This tractate, and some others, were based on lectures delivered to the Members of the College of Preceptors at their monthly meetings. Mr. Ellis took *great* interest in the work of the College. (*Educational Times*, December, 1890.)

"On Scalar and Clinant Algebraical and Coordinate Geometry, introducing a new and more general theory of Analytical Geometry, including the received as a particular case, and explaining 'imaginary points,' 'intersections,' and 'lines.'" [*Royal Soc. Proc.*, Vol. x., 1859–60, pp. 415–426.]

"On an Application of the Theory of Scalar and Clinant Radical Loci." [*Royal Soc. Proc.*, Vol. xi., 1860–62, pp. 141–2; 393–403.]

"On a simple Formula and practical Rule for calculating Heights barometrically without Logarithms." [*Royal Soc. Proc.*, Vol. xii., 1862–3, pp. 511–517.]

"On Stigmatics." [*Brit. Assoc.*, 1864, p. 2.]

"On Plane Stigmatics." [*Royal Soc. Proc.*, Vol. xiv., 1865, pp. 176–8; Vol. xv., 1867, pp. 192–203.]

"On the Corrections for Latitude and Temperature in Barometric Hypsometry, with an improved form of Laplace's formula." [*Royal Soc. Proc.*, Vol. xiv., 1865, pp. 274–288. *Phil. Mag.*, Vol. xxx., 1865, pp. 144–157.]

"On Practical Hypsometry." [*Brit. Assoc.*, 1866, p. 1.]

"Contributions to Formal Logic." [*Royal Soc. Proc.*, 1872, pp. 307–8.]

"On the Algebraical Analogues of Logical Relations." [*Royal Soc. Proc.*, 1873, pp. 497–8.]*

R. T.

Sir J. Cockle furnishes the following list of Errata for insertion here:—

ADDITIONAL ERRATA.

VOL. XVIII.

P. 183, Art. 18, last line. In value of U', *for* u', u *read* w', w.

P. 190, Art. 40, line 2. After -1 *read* $= 0$.

P. 192, Art. 51, line 3, *dele* the last accent.

P. 192, Art. 51, line 6, accent the middle $\dot{Y}$.

VOL. XIX.

P. 265, Art. 113, last line, *for* Y' *read* $\dot{Y}'$.

P. 269, Art. 128, line 8, *for* 90 *read* 110.

P. 275, Art. 153, line 3. In value of μ_2, *for* $-2x\dot{r}$ *read* $-kx\dot{r}$.

P. 276, Art. 157, line 5. In value of M_1, *for* $-2x^2$ *read* $-x^2$.

P. 276, Art. 157, line 7, *for* $-pL$ *read* $-\dot{p}L$.

P. 276, Art. 157, line 8, *for* $-pN$ *read* $-KN$.

ERRATA IN VOL. XX.

P. 10, line 15, *after* "experience" *read* "and he says 'All reasonings concerning matter of fact seem to be founded on the relation of cause and effect.'"

P. 10, line 21, *for ib. read* Inquiry.

* We are indebted in the above notice to an advance copy of the Royal Society's Obituary notice by "W. P."; to an Editorial Notice in the *Academy*; and to Mr. Tristram Ellis.

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Printed by C. F. Hodgson & Son, 2 Newton Street, Holborn W.C.

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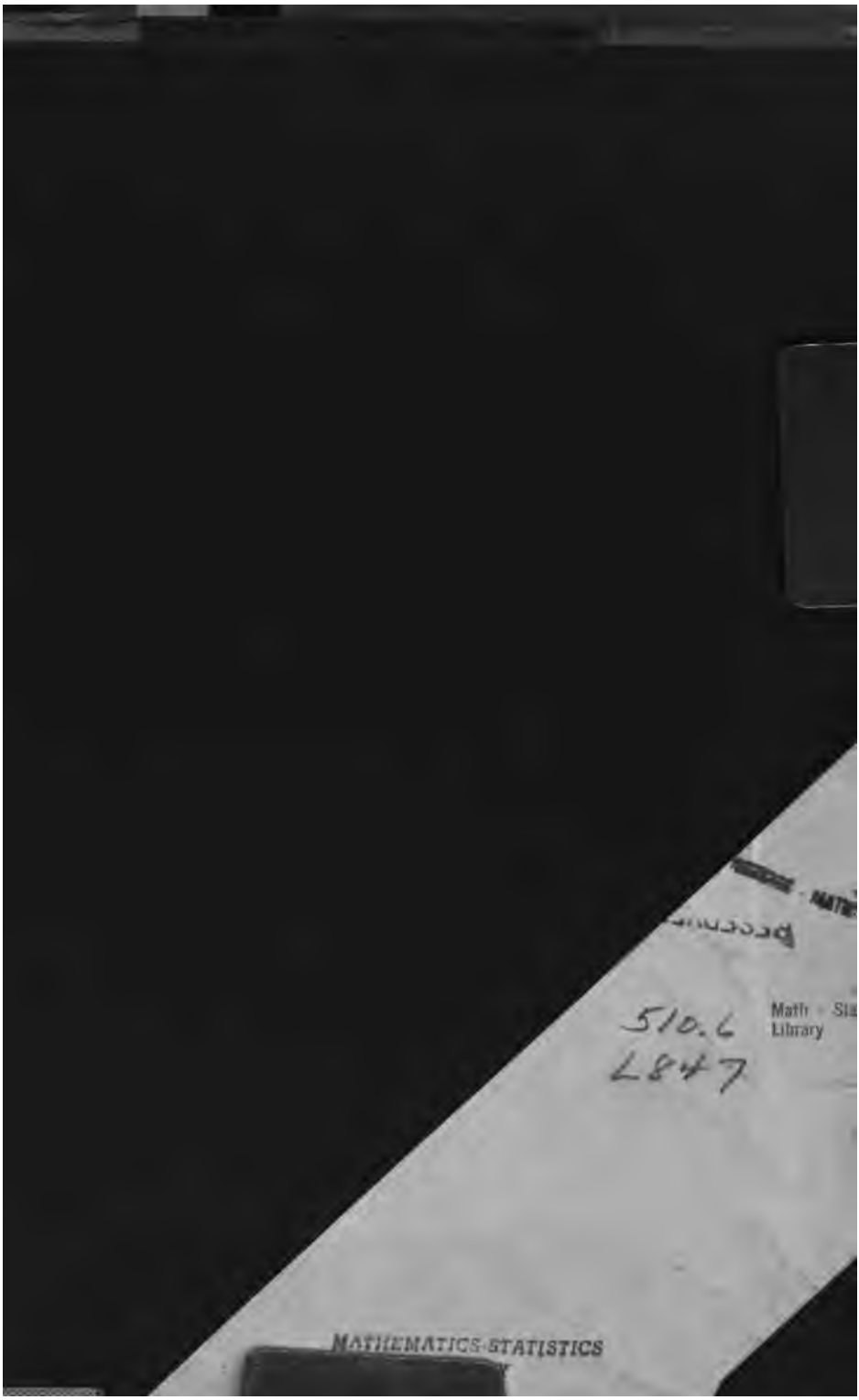
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